2.5 - Zeros of Polynomial Functions

→ To find the zeros of a polynomial, we have usually factored.

it turns out that we can write any n^{th} degree polynomial as the product of n linear factors

$$f(x) = a_n(x-c_1)(x-c_2)\cdots(x-c_n)$$

→ linear factor theorem

where $c_1, c_2, \ldots, c_n$ are complex numbers.

→ The difficult part is actually doing the factoring

Rational Zero Test

If a polynomial $f(x)$ has integer coefficients, every rational zero of f has the form

$$\text{rational zero} = \frac{p}{q}$$

where $\frac{p}{q}$ is in lowest terms and p is a factor of the constant term and q is a factor of the leading coefficient.

EX factor $f(x) = x^3 - 6x^2 + 11x - 6$

→ factors of -6: $\pm 1, \pm 2, \pm 3, \pm 6$

factors of 1: ± 1

all possible combinations of something from the first list over something from the second

So our possible rational zeros are $\pm 1, \pm 2, \pm 3, \pm 6$ (8 options)

we test them out until we find one.

$x=1$:

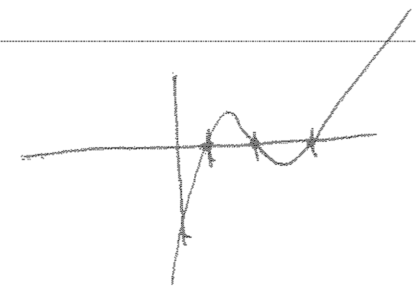
$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & \downarrow & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \underline{0} \end{array}$$

→ we were lucky and found one on our first try.

so $\dfrac{x^3-6x^2+11x-6}{x-1} = x^2-5x+6$

$\Rightarrow f(x) = (x-1)(x^2-5x+6)$

$= (x-1)(x-2)(x-3)$



so zeros are $x=1,2,3$

<u>Ex</u> Find the zeros of $h(t) = t^3+12t^2+21t+10$

factors of 10: $\pm1, \pm2, \pm5, \pm10$

ictors of 1 : ±1

possible rational zeros: $\pm1, \pm2, \pm5, \pm10$

$x=1$:

$$\begin{array}{r|rrrr} 1 & 1 & 12 & 21 & 10 \\ & & 1 & 13 & 33 \\ \hline & 1 & 13 & 33 & 43 \end{array} \to \text{nope!}$$

$$\begin{array}{r|rrrr} -1 & 1 & 12 & 21 & 10 \\ & & -1 & -11 & -10 \\ \hline & 1 & 11 & 10 & 0 \end{array}$$

so $t+1$ is a factor

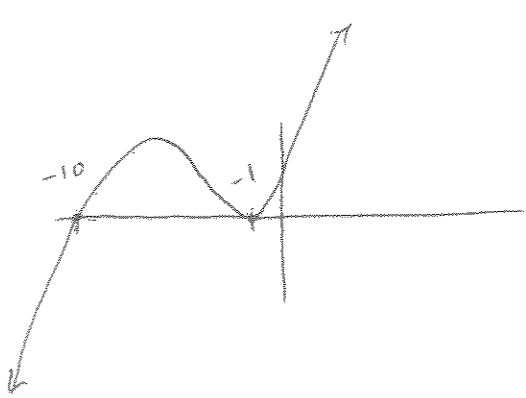
$\dfrac{t^3+12t^2+21t+10}{t+1}$

$= t^2+11t+10$

$f(x) = (t+1)(t^2+11t+10)$

$= (t+1)(t+1)(t+10) = (t+1)^2(t+10)$

zeros: $t=-1,-10$



Ex Find a third degree polynomial with zeros at $x = 1, 3, -2$ and such that $f(2) = 5$

→ use linear factor theorem

$\Rightarrow f(x) = a(x-1)(x-3)(x+2)$

and $5 = a(2-1)(2-3)(2+2)$

$\Rightarrow 5 = a(1)(-1)(4)$

$\Rightarrow 5 = -4a$

$\Rightarrow -\frac{5}{4} = a$

Then $f(x) = -\frac{5}{4}(x-1)(x-3)(x+2)$

→ Sometimes our factorizations involve complex numbers as zeros.

Ex Find the zeros of $f(x) = x^4 + 2x^3 + 10x^2 + 18x + 9$

factors of 9: $\pm 1, \pm 3, \pm 9$

factors of 1 : ± 1

possible rational zeros: $\pm 1, \pm 3, \pm 9$

$x=1$:
$$\begin{array}{r|rrrrr} 1 & 1 & 2 & 10 & 18 & 9 \\ & & 1 & 3 & 13 & 31 \\ \hline & 1 & 3 & 13 & 31 & \underline{40} \end{array}$$
nope

$x=-1$:
$$\begin{array}{r|rrrrr} -1 & 1 & 2 & 10 & 18 & 9 \\ & & -1 & -1 & -9 & -9 \\ \hline & 1 & 1 & 9 & 9 & \underline{0} \end{array}$$

So $f(x) = (x+1)(x^3 + x^2 + 9x + 9)$ → now factor this

possible rational zeros: $\pm 1, \pm 3, \pm 9$

$x=1$ doesn't work.

Try $x=-1$ again:
$$\begin{array}{r|rrrr} -1 & 1 & 1 & 9 & 9 \\ & & -1 & 0 & -9 \\ \hline & 1 & 0 & 9 & \underline{0} \end{array}$$
so $x^3 + x^2 + 9x + 9 = (x+1)(x^2+9)$

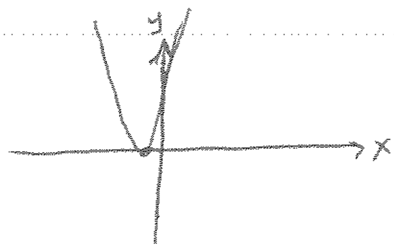
so $f(x) = (x+1)(x+1)(x^2+9)$

$= (x+1)(x+1)(x+3i)(x-3i)$

complex roots always come in pairs like this.

multiplicity 2

so zeros are $x = -1, -3i, 3i$



<u>Ex</u> Solve $2y^4 + 7y^3 - 26y^2 + 23y - 6 = 0$

factors of -6: $\pm 1, \pm 2, \pm 6$

factors of 2: $\pm 1, \pm 2$

possible rational zeros: $1, -1, \frac{1}{2}, -\frac{1}{2}, 2, -2, 6, -6, 3, -3$

Try $x=1$:

$$\begin{array}{r|rrrrr} 1 & 2 & 7 & -26 & 23 & -6 \\ & & 2 & 9 & -17 & 6 \\ \hline & 2 & 9 & -17 & 6 & \underline{0} \end{array}$$

so $y-1$ is a factor

$\Rightarrow (y-1)(2y^3 + 9y^2 - 17y + 6) = 0$

now use this. Same possible rational zeros.

$x=1$:

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -17 & 6 \\ & & 2 & 11 & -6 \\ \hline & 2 & 11 & -6 & \underline{0} \end{array}$$

so $y-1$ is a factor

$\Rightarrow (y-1)(y-1)(2y^2 + 11y - 6) = 0$

$\Rightarrow (y-1)(y-1)(2y-1)(y+6) = 0$

so $\boxed{y = 1, 1/2, -6}$

Ex Find a third degree polynomial with zeros at $x=1$ and $x=2i$, where $f(0)=4$

$\hookrightarrow$ if $x=2i$ is a root, then $x=-2i$ is also a root.

so $f(x) = a(x-1)(x-2i)(x+2i)$

Then $4 = a(0-1)(0-2i)(0+2i)$

$\Rightarrow \quad = a(-1)(-2i)(2i)$

$4 = -4a$

$\Rightarrow -1 = a$

So $f(x) = -(x-1)(x-2i)(x+2i)$

Ex Solve $x^3 - x + 6 = 0$

possible rational zeros: $\pm 1, \pm 2, \pm 6$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -1 & 6 \\ & & 1 & 1 & 0 \\ \hline & 1 & 1 & 0 & \underline{6} \end{array} \quad \text{nope}$$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -1 & 6 \\ & & -1 & 1 & -1 \\ \hline & 1 & -1 & 0 & \underline{5} \end{array} \quad \text{nope}$$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -1 & 6 \\ & & 2 & 4 & 6 \\ \hline & 1 & 2 & 3 & \underline{12} \end{array} \quad \text{nope}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -1 & 6 \\ & & -2 & 4 & -6 \\ \hline & 1 & -2 & 3 & \underline{0} \end{array}$$

can't factor

so $(x+2)(x^2-2x+3) = 0$

$\Rightarrow x+2=0$ or $x^2 \; 2x+3 = 0$

$\Rightarrow \boxed{x=-2}$

$\Rightarrow x^2 - 2x = -3$

$\Rightarrow x^2 - 2x + \left(-\frac{2}{2}\right)^2 = -3 + \left(-\frac{2}{2}\right)^2$

$\Rightarrow (x-1)^2 = -2$

$\Rightarrow x - 1 = \pm\sqrt{-2}$ $\Rightarrow \boxed{x = 1 \pm \sqrt{2}\, i}$