2.3 - Polynomial & Synthetic Division

→ we talked about addition, subtraction, & multiplication of polynomials before. In this section we talk about division & its uses.

→ There are two types of division, polynomial long division and synthetic division.

Polynomial Long Division

↪ Recall long division of numbers

$$\begin{array}{rl}
 & 24853 \text{ r } 15 \\
23 & \overline{)\,571634} \\
 & \underline{46} \\
 & 111 \\
 & \underline{92} \\
 & 196 \\
 & \underline{184} \\
 & 123 \\
 & \underline{115} \\
 & 84 \\
 & \underline{69} \\
 & 15
\end{array}$$

→ Long division of polynomials follows the same pattern

Ex Divide $f(x) = 2x^3 + 3x^2 - 5x - 6$ by $x+1$

$$\begin{array}{rrl}
 & 2x^2 + x - 6 & \rightarrow \text{quotient} \\
\text{divisor} \rightarrow x+1 & \overline{)\,2x^3 + 3x^2 - 5x - 6} & \rightarrow \text{dividend} \\
 & \underline{-(2x^3 + 2x^2)} & \\
 & x^2 - 5x & \\
 & \underline{-(x^2 + x)} & \\
 & -6x - 6 & \\
 & \underline{-(-6x - 6)} & \\
 & 0 &
\end{array}$$

So then $\dfrac{2x^3 + 3x^2 - 5x - 6}{x+1} = 2x^2 + x - 6$

→ Since there was no remainder, it means that $x+1$ is a factor of $2x^3+3x^2-5x-6$, i.e. $2x^3+3x^2-5x-6 = (x+1)(2x^2+x-6)$

$= (x+1)(x+2)(2x-3)$

→ long division helped us factor the polynomial

EX Divide $f(x) = 2x^3+3x^2-5x-6$ by $x-1$

$$
\begin{array}{rl}
 & 2x^2+5x+0 \xrightarrow{\quad} \text{quotient} \\
\text{visor} \to \ x-1 & \overline{)\,2x^3+3x^2-5x-6} \xrightarrow{\quad} \text{dividend} \\
 & -(2x^3-2x^2) \\ \hline
 & 5x^2-5x \\
 & -(5x^2-5x) \\ \hline
 & 0-6 \\
 & -(0-0) \\ \hline
 & -6 \longrightarrow \text{remainder}
\end{array}
$$

Thus
$$\frac{\overset{\text{dividend}}{2x^3+3x^2-5x-6}}{\underset{\text{divisor}}{x-1}} = \overbrace{2x^2+5x}^{\text{quotient}} + \frac{\overset{\text{remainder}}{-6}}{\underset{\text{divisor}}{x-1}}$$

→ $x-1$ is not a factor of $2x^3+3x^2-5x-6$ because the remainder is not zero.

Rather: $2x^3+3x^2-5x-6 = (x-1)(2x^2+5x) - 6$

EX Divide x^3+8 by $x+2$

→ use placeholders for missing terms

$$
\begin{array}{rl}
 & x^2-2x+4 \\
x+2 & \overline{)\,x^3+0x^2+0x+8} \\
 & -(x^3+2x^2) \\ \hline
 & -2x^2+0x \\
 & -(-2x^2-4x) \\ \hline
 & 4x+8 \\
 & -(4x+8) \\ \hline
 & 0
\end{array}
$$

So $\frac{x^3+8}{x+2} = x^2-2x+4$

$\Rightarrow x^3+8 = (x+2)(x^2-2x+4)$

↳ we saw this sum of cubes formula earlier

→ There's a shortcut for dividing polynomials by something of the form $x-k$.

Synthetic Division

Ex Divide $f(x) = x^4 + 3x^3 - x + 7$ by $x+2$

divisor: $x+2$ → coefficients in polynomial

$$\begin{array}{r|rrrrr} -2 & 1 & 3 & 0 & -1 & 7 \\ & \downarrow & \nearrow -2 \downarrow & \nearrow -2 \downarrow & \nearrow 4 \downarrow & \nearrow -6 \\ \hline & 1 & 1 & -2 & +3 & \boxed{1} \end{array}$$

→ remainder

so $$\frac{x^4+3x^3-x+7}{x+2} = x^3 + x^2 - 2x + 3 + \frac{1}{x+2}$$

→ quotient always 1 degree less than divisor

→ One use of synthetic division besides helping us factor is to evaluate polynomials.

Remainder theorem

If a polynomial $f(x)$ is divided by , the remainder is the value of the function at $x=k$. i.e. $r = f(k)$

↳ using the f from our last example,

$$f(-2) = (-2)^4 + 3(-2)^3 - (-2) + 7 = 16 + 3(-8) + 2 + 7 = 16 - 24 + 9 = 1$$

↓ same as remainder

EX Let $f(x) = 2x^3 + 4x^2 - x + 11$. Evaluate $f(3)$, using synthetic division.

$$\begin{array}{r|rrrr} 3 & 2 & 4 & -1 & 11 \\ & \downarrow & 6 & 30 & 87 \\ \hline & 2 & 10 & 29 & \boxed{98} \end{array}$$

So $f(3) = 98$

Also $\dfrac{2x^3+4x^2-x+11}{x-3} = 2x^2 + 10x + 29 + \dfrac{98}{x-3}$

→ we've already seen that division can help us factor.

EX factor $f(x) = 2x^4 - x^3 - 16x^2 - 3x + 18$ given that $(x-1)$ and $(x+2)$ are factors.

→ so $f(x) = (x-1)(x+2)(\text{something})$

↳ use synthetic division

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & -16 & -3 & 18 \\ & \downarrow & 2 & 1 & -15 & -18 \\ \hline & 2 & 1 & -15 & -18 & \boxed{0} \end{array}$$

So $2x^4 - x^3 - 16x^2 - 3x + 18 = (x-1)(2x^3 + x^2 - 15x - 18)$

↳ Again

$$\begin{array}{r|rrrr} -2 & 2 & 1 & -15 & -18 \\ & \downarrow & -4 & 6 & 18 \\ \hline & 2 & -3 & -9 & \boxed{0} \end{array}$$

so $2x^3 + x^2 - 15x - 18 = (x+2)(2x^2 - 3x - 9)$

$= (x+2)(2x + 3)(x - 3)$

Thus $f(x) = (x-1)(x+2)(2x+3)(x-3)$