1.9 - Inverse Functions

→ Recall that with a function, for each input there is one specific output. So if you know the input, you know the output. It doesn't always go the other way though. If you know the output of a function, you can't always be sure what the input was.

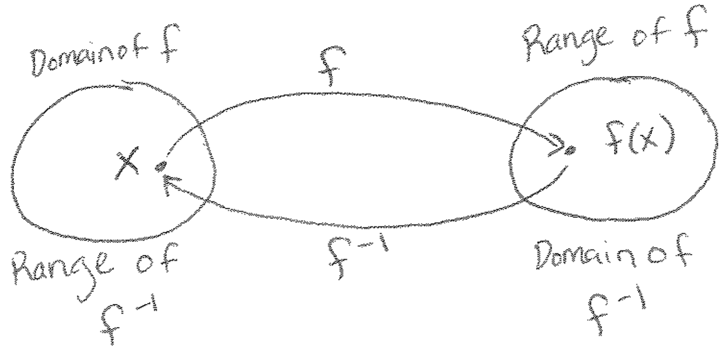
↳ Consider $f(x) = x^2$. If the output is 4, the input was either 2 or -2, but you don't know which it was.

↳ There are some functions, though, where if you know the output, you know exactly what the input must have been.

↳ Consider $f(x) = 2x - 1$. If the output is 5, the input must have been 3.

↳ Functions like this are called invertible. For an invertible function, f, its inverse is called "f inverse" and it is written $f^{-1}(x)$. → inverse, not negative exponent.

↳ Functions and their inverse work like this



→ Inverse functions "undo" each other. That's how we show that two functions are inverses.

→ Two functions f and g are inverses of each other if

$f(g(x)) = x$ and $g(f(x))$.

Ex $f(x) = \frac{1}{x+1}$ and $g(x) = \frac{1}{x} - 1$ are inverses because

$$f(g(x)) = \frac{1}{\left(\frac{1}{x}-1\right)+1} = \frac{1}{\frac{1}{x}-1+1} = \frac{1}{\frac{1}{x}} = 1 \cdot \frac{x}{1} = x$$

$$g(f(x)) = \frac{1}{\frac{1}{x+1}} - 1 = 1 \cdot \frac{x+1}{1} - 1 = x+1-1 = x$$

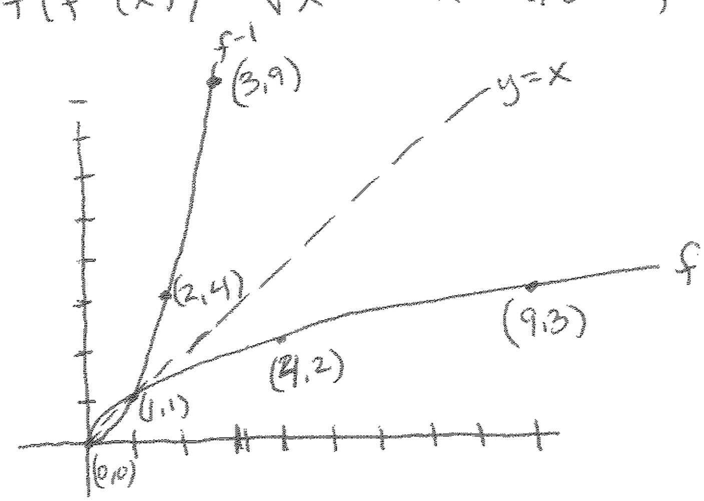
→ Another way of saying the above stetement is that

if $f(x) = y$, then $f^{-1}(y) = x$

↳ functions and their inverses have their x and y values swapped. Graphically, this has interesting implications

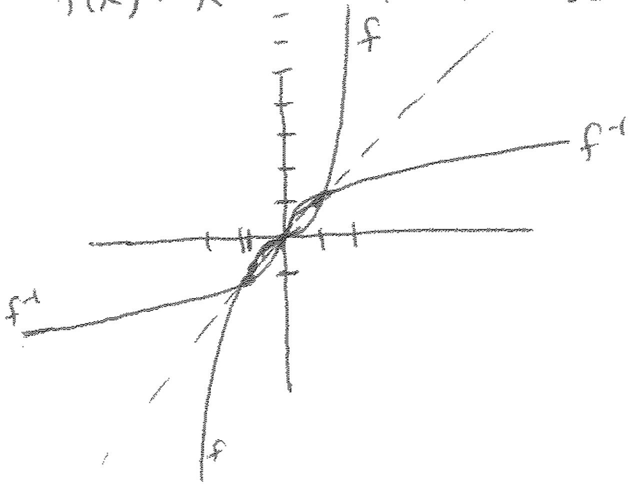
Ex Let $f(x) = \sqrt{x}$. Then $f^{-1}(x) = x^2$; $x \geq 0$

Notice $f(f^{-1}(x)) = \sqrt{x^2} = x$ and $f^{-1}(f(x)) = (\sqrt{x})^2 = x$



↳ The graphs of inverse functions are always reflections across the line $y=x$.

Ex Plot $f(x) = x^3$ and its inverse



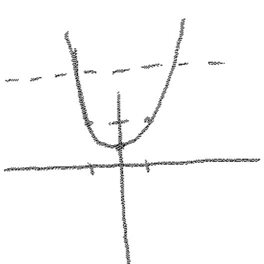
↳ We said earlier that only some functions have inverses. How do we know if a function will have an inverse?

Horizontal line test

A function has an inverse if a horizontal line intersects the graph at most one time. A function that passes the horizontal line test is called one-to-one.

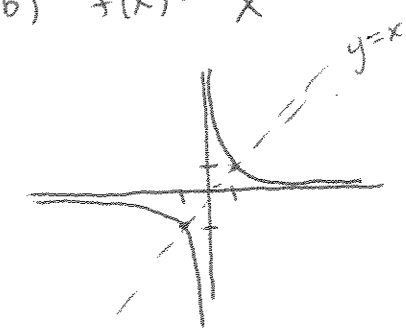
Ex Does it have an inverse?

a) $f(x) = x^2+1$



No! It fails the horizontal line test. The function is not 1-to-1

b) $f(x) = \frac{1}{x}$



yes. It passes the horizontal line test. The function is 1-to-1.

→ the inverse is actually $f^{-1}(x) = \frac{1}{x}$

$$f^{-1}(f(x)) = f(f^{-1}(x)) = \frac{1}{\left(\frac{1}{x}\right)} = 1 \cdot \frac{x}{1} = x$$

↳ we can tell if an inverse exists using the horizontal line test and we can draw the graph of an inverse, but how do we find one algebraically?

↳ We use the fact that inverse functions have their x and y values swapped.

Ex find the inverse of $f(x) = 3x+2$

↳ Does an inverse exist? yes.

write $y = 3x+2$

Now swap x and y and solve for y.

$$x = 3y+2$$
$$\Rightarrow x-2 = 3y$$
$$\Rightarrow \frac{x-2}{3} = y$$

So $f^{-1}(x) = \frac{x-2}{3}$

Ex find inverse of $f(x) = (2x-1)^3+1$

$$y = (2x-1)^3+1$$

swap x + y

$$x = (2y-1)^3+1$$
$$\Rightarrow x-1 = (2y-1)^3$$
$$\Rightarrow (x-1)^{1/3} = 2y-1$$
$$\Rightarrow (x-1)^{1/3}+1 = 2y$$
$$\Rightarrow \frac{(x-1)^{1/3}+1}{2} = y$$

So $f^{-1}(x) = \frac{(x-1)^{1/3}+1}{2}$