9.6 - Applications of Exponentials & Logarithms

→ There are many real-world applications of exponentials and logarithms.

→ we'll look at a specific application; exponential growth / decay

Exponential Growth: Compound interest

→ The difference between simple interest & compound interest is that with simple interest, the amout you make is based on the original amount you invested. with compound interest, the amount you make each time is based on how much is in the account at the beginning of the billing cycle.

→ Compound interest is an example of exponential growth.

→ Suppose you invest some money at an annual interest rate of r (usually given as a percent). Let P_t be the amount of money you have after t years. If you get $\underbrace{\text{paid interest at the end of each year}}_{\text{compounded annually}}$, you have

Time in years	Amount of Money
0	P_0
1	$P_1 = P_0 + rP_0 = P_0(1+r)$
2	$P_2 = P_1 + rP_1 = P_1(1+r) = P_0(1+r)(1+r) = P_0(1+r)^2$
3	$P_3 = P_2 + rP_2 = P_2(1+r) = P_0(1+r)^2(1+r) = P_0(1+r)^3$
⋮	⋮
t	$P_t = P_0(1+r)^t$

→ To get from one year to the next year, you multiply by $(1+r)$

↳ this multiplying by $(1+r)$ is what makes it exponential growth

→ Amount after t years is

$$A(t) = P_0(1+r)^t$$

→ If you invest \$1000 at annual rate of 8%, at the end of 10 years you have

$$A(10) = 1000(1+0.08)^{10}$$
$$= 1000(1.08)^{10}$$
$$\approx 1000(2.15892)$$
$$\approx 2158.92$$

→ Suppose you compound n times a year instead of just once. Then to get from one ~~[illegible]~~ pay time to the next, you multiply by

$\left(1+\frac{r}{n}\right)$ → basically, you split the interest rate into n equal parts and compound n times a year.

→ Amount after t years is given by

$$A(t) = P_0\left(1+\frac{r}{n}\right)^{nt}$$

→ You invest \$1000 in account with 8% annual rate compounded monthly $\Rightarrow n=12$

After 10 years you have

$$A(10) = 1000\left(1+\frac{0.08}{12}\right)^{(12)(10)}$$
$$\approx 1000(1+.00667)^{12(10)}$$
$$\approx 1000(1.00667)^{120}$$
$$\approx 1000(2.21964)$$
$$\approx 2219.64$$

→ \$60.72 more than if you only compound once a year

\$269.76 more after 20 years

\$886.12 more after 30 years

→ simple interest after 10 years; same inital amount & same interest rate.

~~$1,800 → 10 years~~

$2,600 → 20 years

$3,400 → 30 years → $7,548.77 less than the account w/ compound interest compounded monthly

Exponential Decay : Radioactive Decay

Nitrogen 13 is radioactive, and has a half life of about 10 minutes. That means if you have a certain amount of ^{13}N, after 10 minutes, half of it is gone.

→ Suppose you start with C_0 grams of something with a half life of K minutes

time	Amount
0 min	C_0
t = K min	$\frac{1}{2}C_0$
t = 2K min	$\frac{1}{2}\left(\frac{1}{2}C_0\right) = \left(\frac{1}{2}\right)^2 C_0$
t = 3K min	$\frac{1}{2}\left(\frac{1}{2}\right)^2 C_0 = \left(\frac{1}{2}\right)^3 C_0$
⋮	⋮
t = nK min	$\left(\frac{1}{2}\right)^n C_0$

→ In terms of t, the amout you have, $C(t)$ is

$$C(t) = \underbrace{C_0}_{\text{initial amount}}\left(\frac{1}{2}\right)^{t/K}$$

t/K → # of half lives in time t

→ Suppose you start with 32 grams of ^{13}N. How much is left after 30 minutes

$$32 \xrightarrow{10\text{min}} 16 \xrightarrow{10\text{ min}} 8 \xrightarrow{10\text{ min}} 4 \text{ grams}$$

or $$C(30) = 32\left(\frac{1}{2}\right)^{30/10} = 32\left(\frac{1}{2}\right)^3 = 32\left(\frac{1}{8}\right) = 4\text{ grams}$$

→ suppose you start with 10 milligrams of ^{142}Ba → Barium 142 which has a half life of 10.6 minutes. How much do you have after 25 minutes?

→ use equation $C(t) = C_0 \left(\frac{1}{2}\right)^{t/k}$

$$C(25) = 10\left(\tfrac{1}{2}\right)^{25/10.6}$$
$$\approx 10\left(\tfrac{1}{2}\right)^{2.358}$$
$$\approx 10(.194995)$$
$$\approx 1.94995 \text{ mg}$$

Ex you invest P_0 dollars at an annual rate of 7% compounded daily
How long does it take your money to triple?

→ when your money has tripled, you have $3P_0$. So then

use: $A(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$

$$3P_0 = P_0\left(1 + \frac{0.07}{365}\right)^{365t}$$
$$\Rightarrow 3 = \left(\frac{365}{365} + \frac{0.07}{365}\right)^{365t}$$
$$\Rightarrow 3 = \left(\frac{365.07}{365}\right)^{365t}$$
$$\Rightarrow \ln 3 = \ln\left(\frac{365.07}{365}\right)^{365t}$$
$$\Rightarrow \ln 3 = (365)(t)\ln\left(\frac{365.07}{365}\right)$$
$$\Rightarrow \frac{\ln 3}{365\ln\left(\frac{365.07}{365}\right)} = t \qquad t \approx 15.696 \text{ years}$$

→ It doesn't matter how much you invest initially, it always takes the same amount of time for that initial investment to triple.