9.2 - Composite & Inverse Functions

→ we've seen composition of functions a few times in the homework.

- If we have some function $f(x)$ and some other function $g(x)$, the composition of f and g, written

$(f \circ g)(x)$ is $f(g(x))$

↳ This means every time you see an x in $f(x)$, substitute in the expression $g(x)$.

Similarly $(g \circ f)(x) = g(f(x))$. → substitute the expression $f(x)$ into the x's in g.

Ex $f(x) = x-5$, $g(x) = 3x+2$

a) $(f \circ g)(x) = f(g(x))$
$= f(3x+2)$
$= (3x+2) - 5$
$= 3x-3$

b) $(g \circ f)(x) = g(f(x))$
$= g(x-5)$
$= 3(x-5)+2$
$= 3x-15+2$
$= 3x-13$

c) $(f \circ g)(2) = 3(2)-3 = 3$

or $(f \circ g)(2) = f(g(2))$

$g(2) = 3(2)+2 = 8$ and $f(8) = 8-5 = 3$

so $(f \circ g)(2) = f(g(2)) = f(8) = 3$

d) $(g \circ f)(2) = 3(2) - 13 = -7$

or $(g \circ f)(2) = g(f(2))$

$f(2) = 2-5 = -3$, $g(-3) = 3(-3)+2 = -7$

so $(g \circ f)(2) = g(f(2)) = g(-3) = -7$

Note: $(f \circ g)(x) \neq (g \circ f)(x)$ in general

Ex $f = \{(-2,3), (-1,1), (0,0), (1,-1), (2,-3)\}$

$g = \{(-3,1), (-1,-2), (0,2), (2,2), (3,1)\}$

a) $f(2) = -3$

) $g(-3) = 1$

) $g(f(2)) = g(-3) = 1$

Ex $f(x) = 2x-1$, $g(x) = \frac{1}{2}(x+1)$

a) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{2}(x+1)\right) = 2\left(\frac{1}{2}(x+1)\right) - 1 = x+1-1 = x$

b) $(g \circ f)(x) = g(f(x)) = g(2x-1) = \frac{1}{2}((2x-1)+1) = \frac{1}{2}(2x) = x$

Notice: $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

↳ $f(x)$ and $g(x)$ are called inverses of each other because the above line is true.

→ Inverse functions kind of undo each other.

Ex a) $f(2) = 2(2) - 1 = 3$

b) $g(3) = \frac{1}{2}(3+1) = 2$

inverse, not exponent

→ Since $g(x)$ is the inverse of f, we say $g(x) = f^{-1}(x)$

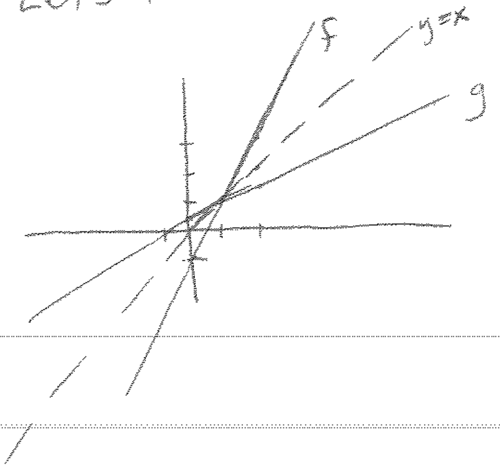
→ This is terrible notation! But it's what we use.

→ In general $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

→ This is how we test to see if functions are inverses.

→ Let's look at f and g graphically.

$g(x) = \frac{1}{2}x + \frac{1}{2}$

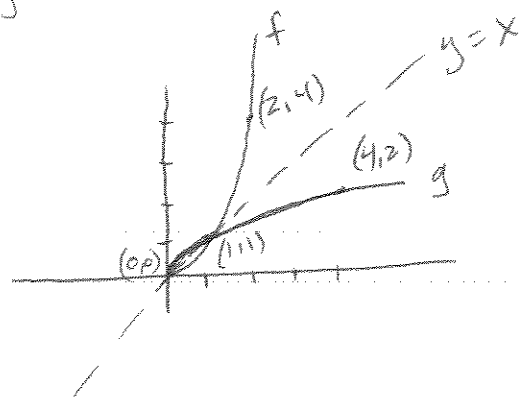


→ f and g are reflections across $y=x$

Ex consider $f(x) = x^2$ with $x \geq 0$ and $g(x) = \sqrt{x}$

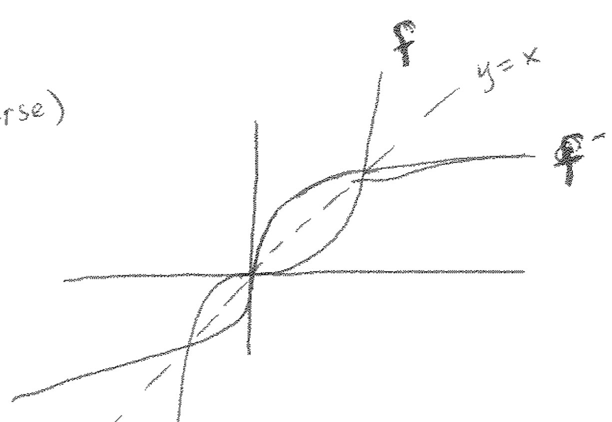
$f(g(x)) = (\sqrt{x})^2 = x$ and $g(f(x)) = \sqrt{x^2} = x$

so f and g are inverses.



→ graphs of inverse functions are reflections across the line $y=x$.

→ Suppose we're given the graph of f. We can draw the graph of f^{-1}
(f inverse)



→ How do we know if an inverse function exists?

→ Remember the vertical line test for functions

↳ A graph is the graph of a function if a vertical line only touches the graph at most one time.

↳ This means that for each input, there is only 1 output

→ what do we get if we reflect a vertical line across the line $y=x$?

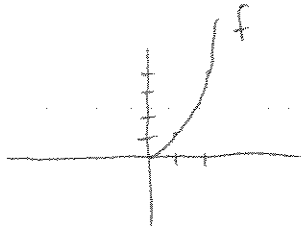
↳ A horizontal line

↳ A function f has an inverse f^{-1} if its graph passes the horizontal line test.

↳ A horizontal line only touches the graph once.

↳ A function that passes the horizontal line test is called 1-to-1

<u>Ex</u> $f(x) = x^2$ with $x \geq 0$

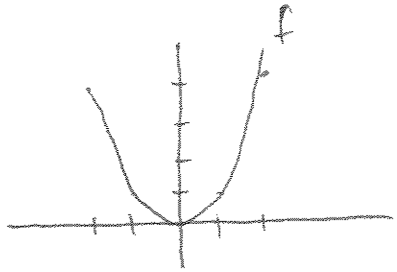


→ passes the horizontal line test.

↳ f^{-1} exists.

↳ we saw that $f^{-1}(x) = \sqrt{x}$

<u>Ex</u> suppose $g(x) = x^2$ (no restriction on x)



→ fails horizontal line test

→ inverse doesn't exist

→ So how do we find an inverse algebraically?

↳ reflecting a graph across $y=x$ is just swapping the x and y values.

→ To find an inverse, we solve the function for x, swap the x's and y's, and call the new thing $f^{-1}(x)$.

Ex $f(x) = 2x-1$

write: $y = 2x-1$

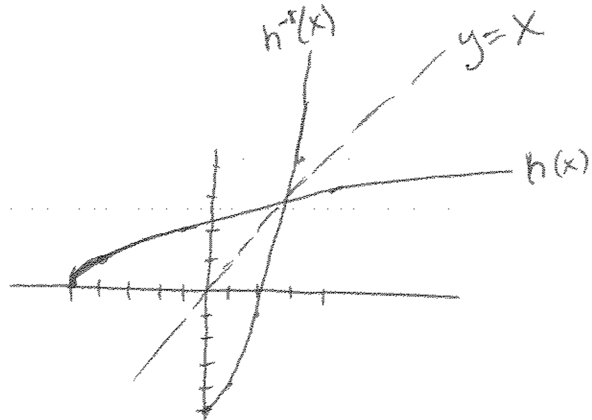
solve for x: $y+1 = 2x$ ⇒ $\frac{y+1}{2} = x$

swap x and y: $\frac{x+1}{2} = y$

call y, $f^{-1}(x)$: $f^{-1}(x) = \frac{x+1}{2}$

→ we've already seen that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Ex $h(x) = \sqrt{x+5}$



$y = \sqrt{x+5}$

⇒ $y^2 = x+5$; $y \geq 0$

⇒ $y^2 - 5 = x$; $y \geq 0$

so $x^2 - 5 = y$; $x \geq 0$

then $f^{-1}(x) = x^2 - 5$; $x \geq 0$

Ex $f(x) = x^2$ → we know f^{-1} doesn't exist. Try anyway

$y = x^2$

$\pm\sqrt{y} = x$

~~$y = \pm\sqrt{x}$~~

$\pm\sqrt{x} = y$

$f^{-1}(x) = \pm\sqrt{x}$ → 2 outputs for each input, so this isn't a function!

Supplementary Problems:

1, 3, 5, 9, 15, 35, 37, 39, 41, 43, 45, 49, 53, 59, 63, 67, 73, 75, 77, 79, 81, 83