8.2 – 8.3 → Completing the square & the quadratic Formula

> Completing the square & the quadratic formula are two related methods to solve quadratic equations.
> They both work to solve <u>any</u> quadratic equation.

Completing the square

Recall: $(x-r)^2 = (x-r)(x-r) = x^2 - rx - rx + r^2 = x^2 - 2rx + r^2$

and $(x+r)^2 = x^2 + 2rx + r^2$

so then $x^2 + 6x + 9 = 0 \Rightarrow (x+3)^2 = 0 \rightarrow x = -3$

what about $x^2 + 6x + 7 = 0$?

→ It would be nice if the 7 were a 9 instead.

→ Add 2 to both sides & make it so!

$$x^2 + 6x + 9 = 2$$
$$\Rightarrow (x+3)^2 = 2$$
$$\Rightarrow x + 3 = \pm\sqrt{2}$$
$$\Rightarrow \boxed{x = -3 \pm \sqrt{2}}$$

Completing the square is the process of making the polynomial into a perfect square so we can solve using the square root property.

> There is a general way we can do this.

Leading coefficient 1.

Ex Solve $y^2+8y+9=0$

~~Ste~~ → move the constant term to the other side

$$y^2+8y=-9$$

→ divide coefficient of y by 2, square it, and add the square to both sides.

$$y^2+8y+\left(\frac{8}{2}\right)^2=-9+\left(\frac{8}{2}\right)^2$$

$$\Rightarrow \underbrace{y^2+2\cdot 4\cdot y+4^2}_{\text{perfect square}}=-9+4^2$$

$$\Rightarrow (y+4)^2=7$$

$$\Rightarrow y+4=\pm\sqrt{7}$$

$$\Rightarrow \boxed{y=-4\pm\sqrt{7}}$$

we can also get complex solutions

Ex $z^2-6z+18=0$

$$z^2-6z=-18$$

$$z^2-6z+(-3)^2=-18+(-3)^2$$

$$(z-3)^2=-9$$

$$z-3=$$

$$\Rightarrow z-3=\pm 3i$$

$$\rightarrow \boxed{z=3\pm 3i}$$

complex solutions always come in complex conjugate pairs!

→ what if the leading coefficient isn't 1?

→ start by dividing everything by the leading coefficient and then proceed as before

EX $-3x^2 + 24x + 5 = 0$ → Divide everything by -3

$x^2 - 8x - \frac{5}{3} = 0$

$x^2 - 8x = \frac{5}{3}$

$x^2 - 8x + (-4)^2 = \frac{5}{3} + (-4)^2$

$(x-4)^2 = \frac{5}{3} + 16$

$(x-4)^2 = \frac{5}{3} + \frac{48}{3} = \frac{53}{3}$

$x - 4 = \pm\sqrt{\frac{53}{3}}$

$x - 4 = \pm\frac{\sqrt{159}}{3}$

$x = -4 \pm \frac{\sqrt{159}}{3}$

EX $4z^2 - 3z + 2 = 0$ $\qquad -\frac{3}{4} \div 2 = \frac{-3}{4} \cdot \frac{1}{2} = \frac{-3}{8}$

$z^2 - \frac{3}{4}z = -1/2$

$z^2 - \frac{3}{4}z + \left(\frac{-3}{8}\right)^2 = \frac{-1}{2} + \left(-\frac{3}{8}\right)^2$

$\left(z - \frac{3}{8}\right)^2 = \frac{-1}{2} + \frac{9}{64}$

$\left(z - \frac{3}{8}\right)^2 = \frac{-32}{64} + \frac{9}{64} = \frac{-23}{64}$

$z - \frac{3}{8} = \pm\sqrt{\frac{-23}{64}} = \frac{\sqrt{23}}{8}i$

$$\boxed{z = \frac{3}{8} \pm \frac{\sqrt{23}}{8}i}$$

Quadratic Formula

→ Suppose we wanted to solve a general quadratic equation by completing the square.

$$ax^2+bx+c=0$$

$$\Rightarrow x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$\Rightarrow x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{-c}{a}+\frac{b^2}{4a^2}$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{-4ca}{4a^2}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \boxed{x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}} \quad \text{or} \quad \boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

→ This is the quadratic formula. We can use it to solve any quadratic equation

→ I recommend completing the square for 2 reasons

1) People tend to make mistakes with the quad. formula

2) we use processes similar to completing the square in other situations too, so its a good tool to use.

• we'll do all the examples of completing the square but using the quadratic formula.

Ex $x^2+6x+9=0$ $a=1,\ b=6,\ c=9$

$$x=\frac{-6\pm\sqrt{36-4(1)(9)}}{2(1)}=\frac{-6\pm\sqrt{36-36}}{2}=\frac{-6\pm\sqrt{0}}{2}=-3$$

one repeated solution

Ex $x^2+6x+7=0$ $a=1,\ b=6,\ c=7$

$$x=\frac{-6\pm\sqrt{36-4(1)(7)}}{2(1)}=\frac{-6\pm\sqrt{36-28}}{2}=\frac{-6\pm\sqrt{8}}{2}$$

$$=\frac{-6\pm2\sqrt{2}}{2}$$

$$=\frac{-6}{2}\pm\frac{2\sqrt{2}}{2}$$

$$=-3\pm\sqrt{2}$$

Ex $y^2+8y+9=0$

$$y=\frac{-8\pm\sqrt{8^2-4(1)(9)}}{2(1)}=\frac{-8\pm\sqrt{64-36}}{2}=\frac{-8\pm\sqrt{28}}{2}$$

$$=\frac{-8\pm2\sqrt{7}}{2}$$

$$=-\frac{8}{2}\pm\frac{2\sqrt{7}}{2}$$

$$=-4\pm\sqrt{7}$$

Ex $z^2 - 6z + 18 = 0$

$$\frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(18)}}{2(1)} = \frac{6 \pm \sqrt{36 - 72}}{2} = \frac{6 \pm \sqrt{-36}}{2}$$

$$= \frac{6 \pm 6i}{2}$$

$$= 3 \pm 3i$$

Ex $-3x^2 + 24x + 5 = 0$

$$x = \frac{-24 \pm \sqrt{24^2 - 4(-3)(5)}}{2(-3)} = \frac{-24 \pm \sqrt{576 + 60}}{-6}$$

$$= \frac{-24 \pm \sqrt{636}}{-6}$$

$$= \frac{-24 \pm \sqrt{4 \cdot 159}}{-6}$$

$$= \frac{-24 \pm 2\sqrt{159}}{-6}$$

$$= -4 \pm \frac{\sqrt{159}}{3}$$

Ex $4z^2 - 3z + 2 = 0$

$$z = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(2)}}{2(4)} = \frac{3 \pm \sqrt{9 - 32}}{8}$$

$$= \frac{3 \pm \sqrt{-23}}{8}$$

$$= \frac{3 \pm \sqrt{23}\, i}{8} = \frac{3}{8} \pm \frac{\sqrt{23}}{8} i$$

Supplementary Problems

8.2 → PP 525 - 528

1, 3, 5, 33, 35, 37, 45, 49, 51, 59, 63, 65, 89, 91, 93

8.3 → PP 534 - 537

15, 17, 21, 23, 25, 35, 49, 51, 53, 55, 57, 59, 63, 97, 99, 103