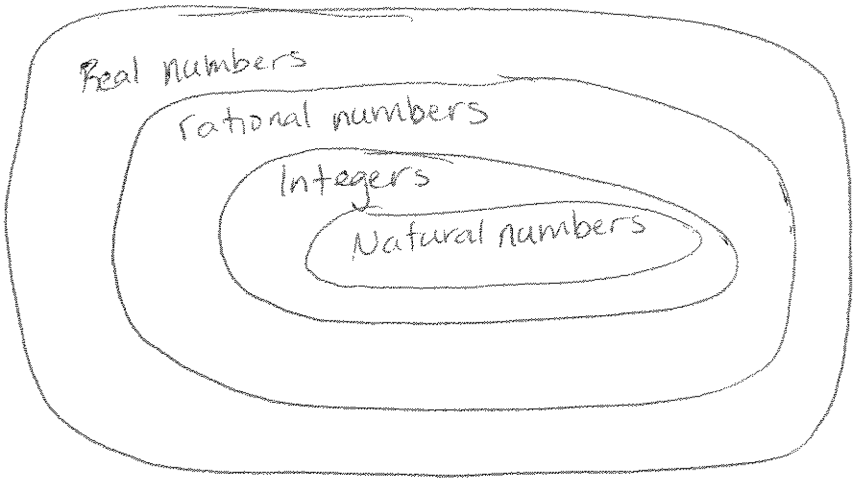
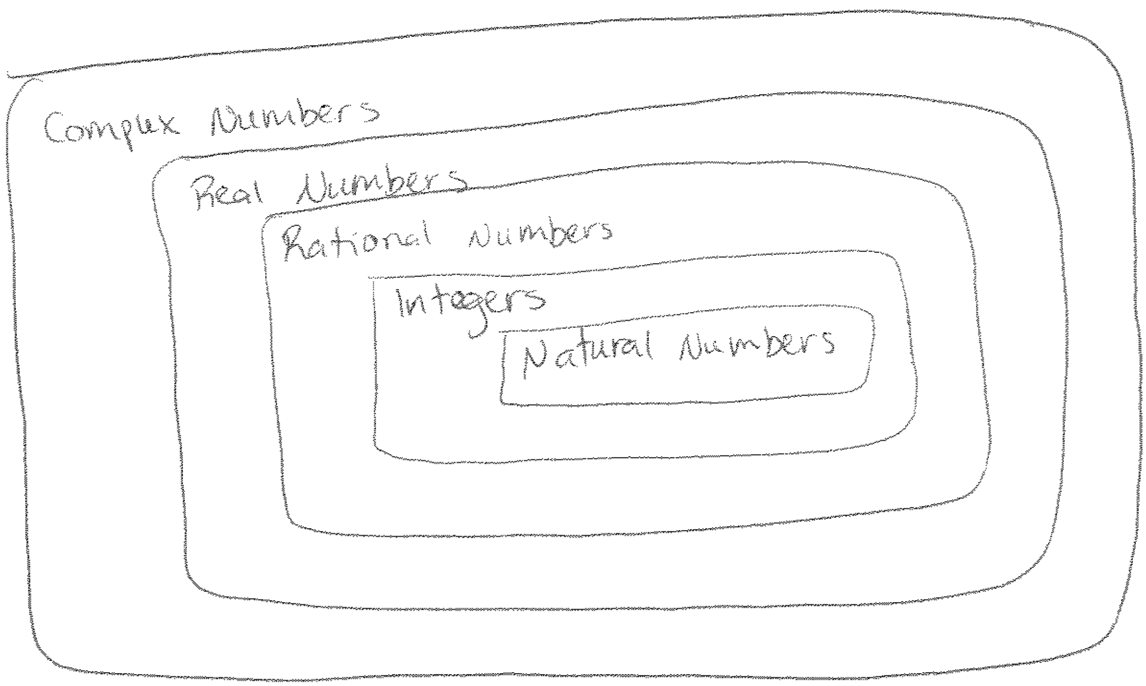
7.6 - Complex Numbers

→ Recall the following picture from way back in section 1.1



→ we've worked with real numbers for almost all of the course

→ In this section, we consider a larger set of numbers called complex numbers.



→ we've frequently come across the problem of taking the square root of a negative number. we said there was no real result.

→ To get around this inconvenience, mathematicians came up with the number i defined as

$$\sqrt{-1} = i \quad \text{or} \quad i^2 = -1$$

- Idea was formalized in the late 1500's and was not widely accepted initially in the same way that the number zero and the negative numbers were not accepted initially
- Complex numbers have wide modern applications in higher mathematics and physics.
- for our uses, we'll use complex numbers to take the square root of a negative number. with this, we can solve any quadratic equation, $ax^2+bx+c=0$
- in practice, we'll treat i as a variable except we'll ~~[illegible]~~ remember that $i^2=-1$

X
a) $\sqrt{-16} = \sqrt{16 \cdot i^2} = 4i$

b) $\sqrt{-32} = \sqrt{16 \cdot 2 \cdot i^2} = 4i\sqrt{2}$

c) $\sqrt{\frac{-16}{36}} = \frac{\sqrt{16i^2}}{\sqrt{36}} = \frac{4i}{6} = \frac{2i}{3}$

c) $\sqrt{-9} + \sqrt{-49} = \sqrt{9i^2} + \sqrt{49i^2} = 3i + 7i = 10i$

Complex numbers:

> Something like $2+3i$ is called a complex number.

A complex number is standard form looks like

$$\underset{\text{real part}}{a} + \underset{\text{imaginary part}}{bi}$$

where a and b are real numbers

if $a=0$, the complex number is called purely imaginary

> 2 complex #'s equal if real & imaginary parts are equal.

→ We can add, subtract, multiply, and divide complex numbers.

→ Since $i = \sqrt{-1}$ is a radical, we pretty much follow the rules for radicals.

Ex a) $(\underset{\text{real part}}{3} + \underset{\text{imaginary part}}{2i}) + (4-7i) = 3+4 + (2-7)i$

$= 7 - 5i$ → combine real parts & imaginary parts

→ FOIL

b) $(1+2i)(2-i) = 2 - i + 4i - 2i^2$

$= 2 + 3i + 2$

$= 4 + 3i$

c) $(1-i)\sqrt{-9} = (1-i)\underbrace{3i}_{\text{purely imaginary}} = 3i - 3i^2 = \cancel{3i+3} \; 3i+3 = \underbrace{3+3i}_{\text{standard form}}$

→ Remember how we could multiply a radical expression like $4+\sqrt{3}$ by its conjugate $4-\sqrt{3}$ and get a result

$(4+\sqrt{3})(4-\sqrt{3}) = 16 - 4\sqrt{3} + 4\sqrt{3} - 3 = 13$ which was no longer a radical.

→ We can do something similar with complex numbers.

Notice: $(2+3i)(2-3i) = 4 - 6i + 6i - 9i^2 = 4 + 9 = 13$

A complex number $a+bi$ has a complex conjugate $a-bi$.

And $(a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2$

↳ The product of complex conjugates is always a real number.

→ we use this fact to write quotients of complex numbers in standard form the same way we rationalized the denominators of radicals.

Ex $\dfrac{3-i}{-2+4i}$ → This is not in standard form!
we can't just divide real parts + imaginary parts to get
$-\frac{3}{2}-\frac{1}{4}$ or $\frac{-3}{2}-\frac{1}{4}i$

→ we must multiply both the numerator + denominator by the complex conjugate of the denominator.

$$\frac{3-i}{-2+4i}\cdot\frac{-2-4i}{-2-4i}=\frac{-6-12i+2i+4i^2}{4+8i-8i-16i^2}=\frac{-6-10i-4}{4+16}=\frac{-10-10i}{20}$$

$$=\frac{10(-1-i)}{20}$$

$$=\frac{-1-i}{2}$$

$$=\underbrace{-\frac{1}{2}-\frac{1}{2}i}_{\text{standard form}}$$

real part imaginary part

Ex $$\frac{2-i}{4i}=\frac{2-i}{4i}\cdot\frac{-4i}{-4i}=\frac{-8i+4i^2}{-16i^2}$$

$$=\frac{-4-8i}{16}=\frac{4(-1-2i)}{16}=\frac{-1-2i}{4}=\underbrace{\frac{-1}{4}-\frac{1}{2}i}_{\text{standard form}}$$

our biggest use for complex numbers is to solve quadratic equations.

Ex $x^2-4x+5=0$ has no real solutions

But, $2+i$ is a solution (so is $2-i$)

Notice:

$$
\begin{aligned}
(2+i)^2 - 4(2+i) + 5 &= (2+i)(2+i) - 4(2+i) + 5 \\
&= 4 + 2i + 2i + i^2 - 8 - 4i + 5 \\
&= 4 + 4i - 1 - 8 - 4i + 5 \\
&= 0
\end{aligned}
$$

→ we'll spend all of chapter 8 working with quadratic equations.

Supplementary Problems: pp. 499-501

1, 3, 5, 13, 19, 21, 25, 43, 45, 49, 55, 57, 69, 71, 75, 89, 91, 123, 125, 127

~~[illegible]~~ 139, 141, 145, 147