6.5 Dividing Polynomials and Synthetic Division

→ We'll talk about two different ways of dividing polynomials. The first is long division, the second is synthetic division.

→ Dividing two ~~numbers~~ integers gives a rational number

→ Dividing two polynomials gives a rational expression.

→ Let's review some terminology using integers.

$$67 \div 6 = 11 \quad R\ 1$$

dividend divisor Quotient remainder

We could also write this

remainder

dividend → $\dfrac{67}{6} = 11 + \dfrac{1}{6}$ ← divisor or $67 = 11 \cdot 6 + 1$

divisor →

Quotient

Similarly, for polynomials, we have for example

remainder

dividend → $\dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$ or $x+2 = 1(x+1) + 1$

divisor →

Quotient divisor

→ But how do we find the quotient and remainder?

↳ with integers, we can use long division

For example

$$\begin{array}{rrl}
 & 235 & \leftarrow \text{quotient} \\
\text{divisor} \longrightarrow & 28\overline{)6584} & \longrightarrow \text{dividend} \\
 & 56 & \\ \cline{2-2}
 & 98 & \\
 & 84 & \\ \cline{2-2}
 & 144 & \\
 & 140 & \\ \cline{2-2}
 & 4 & \leftarrow \text{remainder}
\end{array}$$

so $6584 \div 28 = 235 \ R4$ or $\dfrac{6584}{28} = 235 + \dfrac{4}{28}$

→ we can also do long division with polynomials

Ex

$$\begin{array}{rrl}
 & x + 3 & \leftarrow \text{quotient} \\
\text{divisor} \rightarrow & x-1\overline{)x^2+2x+4} & \leftarrow \text{dividend} \\
 & -(x^2 - x) & \\ \cline{2-2}
 & 3x+4 & \\
 & -(3x-3) & \\ \cline{2-2}
 & 7 & \leftarrow \text{remainder}
\end{array}$$

so then $\dfrac{x^2+2x+4}{x-1} = x+3 + \dfrac{7}{x-1}$ or $x^2+2x+4 = (x+3)(x-1) + 7$

Ex

$$\begin{array}{rl}
 x - 6 & \\
x+1\overline{)x^2-5x-6} & \\
-(x^2 + x) & \\ \hline
-6x-6 & \\
-(-6x-6) & \\ \hline
0 & \rightarrow \text{remainder zero}
\end{array}$$

notice: $x^2-5x-6 = (x+1)(x-6)$

so $\dfrac{(x+1)(x-6)}{x+1} = x-6;\ x \neq -$

→ This illustrates an important idea, we can use long division to help us factor

Ex Factor $x^3 - x^2 - 10x - 8$ given that $(x+1)$ is a factor.

$$
\begin{array}{r}
x^2 - 2x - 8 \\
x+1 \,\big)\, \overline{x^3 - x^2 - 10x - 8} \\
-(x^3 + x^2) \\
\hline
-2x^2 - 10x \\
-(-2x^2 - 2x) \\
\hline
-8x - 8 \\
-(-8x - 8) \\
\hline
0
\end{array}
$$

So then $\dfrac{x^3 - x^2 - 10x - 8}{x+1} = x^2 - 2x - 8 \Rightarrow x^3 - x^2 - 10x - 8 = (x+1)(x^2 - 2x - 8)$

$$= (x+1)(x+2)(x-4)$$

→ Make sure things are in standard form before performing long division.

Ex $2x + 3 + 8x^2 \div -1 + 4x$

Standard form: $8x^2 + 2x + 3 \div 4x - 1$

$$
\begin{array}{r}
2x + 1 \\
4x-1 \,\big)\, \overline{8x^2 + 2x + 3} \\
-(4x^2 - 2x) \\
\hline
4x + 3 \\
-(4x - 1) \\
\hline
4
\end{array}
$$

so $\dfrac{8x^2 + 2x + 3}{4x - 1} = 2x + 1 + \dfrac{4}{4x-1}$

→ We can divide by things that aren't linear

Ex $(2x^3+2x^2-2x-15) \div (2x^2+4x+5)$

$$
\begin{array}{r l}
 & \quad\quad\quad\quad\quad\; x - 1 \\
2x^2+4x+5 \,\big) & 2x^3+2x^2-2x-15 \\
 & -(2x^3+4x^2+5x) \\
\hline
 & \quad\quad -2x^2-7x-15 \\
 & \quad -(-2x^2-4x-5) \\
\hline
 & \quad\quad\quad\; -3x-10
\end{array}
$$

so $\dfrac{2x^3+2x^2-2x-15}{2x^2+4x+5} = x-1+\dfrac{-3x-10}{2x^2+4x+5}$

→ sometimes terms are missing. We add the missing terms with coefficients of zero as placeholders.

Ex $\dfrac{y^2+8}{y+2}$

$$
\begin{array}{r l}
 & \quad\quad\quad y-2 \\
y+2 \,\big) & y^2+0y+8 \\
 & -(y^2+2y) \\
\hline
 & \quad\; -2y+8 \\
 & \; -(-2y-4) \\
\hline
 & \quad\quad\quad 12
\end{array}
$$

So $\dfrac{y^2+8}{y+2} = y-2+\dfrac{12}{y+2}$

→ The coefficients won't always be ~~these~~ integers

Ex $(x^2-1) \div (7x+3)$

$$
\begin{array}{r|l}
 & \frac{1}{7}x - \frac{3}{49} \\
\hline
7x+3 & x^2 + 0x - 1 \\
 & -(x^2 + \frac{3}{7}x) \\
\hline
 & -\frac{3}{7}x - 1 \\
 & -\left(-\frac{3}{7}x - \frac{9}{49}\right) \\
\hline
 & -\frac{40}{49}
\end{array}
$$

→ when we're dividing by something of the form $x-a$ where a is some number, There's a shorter way to do this called synthetic division.

Ex $x^2+10x-9 \div x-3$

Long division:

$$
\begin{array}{r|l}
 & x + 13 \\
\hline
x-3 & x^2+10x-9 \\
 & -(x^2-3x) \\
\hline
 & 13x-9 \\
 & -(13x-39) \\
\hline
 & 30
\end{array}
$$

Synthetic division:

$$
\begin{array}{r|rrr}
3 & 1 & 10 & -9 \\
 & \downarrow & 3 & 39 \\
\hline
 & \underbrace{1 \quad 13}_{\text{coefficients of quotient}} & & \underset{\text{remainder}}{30}
\end{array}
$$

→ coefficients of dividend

divisor (notice that the sign is opposite)

→ Bring down 1st #
then multiply diagonals
& add verticals

Notice: if we plug $x=3$ into $x^2+10x-9$ we have

$3^2+10(3)-9 = 9+30-9 = 30$ → same as remainder

Ex Evaluate $4x^4 + x^2 + 8x - 2$ at $x = 6$

→ Anyone know 6^4? We could do this using synthetic division (or long division by $(x-6)$)

placeholder for x^3 term

number we evaluate at →

$$
\begin{array}{r|rrrrr}
6 & 4 & 0 & 1 & 8 & -2 \\
 & \downarrow & 24 & 144 & 870 & 5268 \\
\hline
 & 4 & 24 & 145 & 878 & \boxed{5266}
\end{array}
\quad \leftarrow \text{coefficients}
$$

We also know that

$$\frac{4x^4 + x^2 + 8x - 2}{x-6} = 4x^3 + 24x^2 + 145x + 878 + \frac{5266}{x-6}$$

Ex Evaluate $x^3 + x^2 - 32x - 60$ at $x = -5$

$$
\begin{array}{r|rrrr}
-5 & 1 & 1 & -32 & -60 \\
 & & -5 & 20 & 60 \\
\hline
 & 1 & -4 & -12 & \boxed{0}
\end{array}
$$

→ The polynomial is zero when $x = -5$. That means $x+5$ must be a factor!

So $x^3 + x^2 - 32x - 60 = (x+5)(x^2 - 4x - 12)$

$$= (x+5)(x-6)(x+2)$$

→ Remember, synthetic division only works to divide by something of the form $x - a$.

Supplementary Problems: pp 415-418

~~[illegible]~~

7, 9, 11, 19, 21, 25, 29, 33, 37, 41, 49, 51, 53, 63, 65, 69, 73, 75, 79, 91