4.3 - Linear Systems in Three Variables

→ We've been working with systems that had 2 linear equations and 2 variables. Now we step up to 3 linear equations and 3 variables.

Consider the systems

$$\begin{aligned} x+y+z&=2 \\ -x+3y+2z&=8 \\ 4x+y\quad&=4 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x+y+z&=2 \\ 4y+3z&=10 \\ z&=-2 \end{aligned}$$

→ which is easier to solve?

↳ Clearly, the one on the right.

$\boxed{z=-2}$, so $4y+3(-2)=10 \Rightarrow 4y-6=10 \Rightarrow 4y=16 \Rightarrow \boxed{y=4}$

Then $x+4-2=2 \Rightarrow x+2=2 \Rightarrow \boxed{x=0}$

→ the system on the right is in a form called row echelon form, and solving like we just did is called back substitution.

→ Actually, the system on the left and the system on the right have the same solution. How do we make the one on the left look like the one on the right?

↳ The process is called Gaussian Elimination, and it's basically a repeated application of ~~the~~ the method of elimination that we learned last section.

→ Let's look back at the system on the left

$$\begin{cases} x + y + z = 2 \\ -x + 3y + 2z = 8 \\ 4x + y \quad = 4 \end{cases}$$

$$\begin{cases} x + y + z = 2 & \\ \quad 4y + 3z = 10 & \rightarrow (1) + (2) \\ \quad 3y + 4z = 4 & \rightarrow 4 \cdot (1) - 3 \end{cases}$$

$$\begin{cases} x + y + z = 2 & \\ \quad 4y + 3z = 10 & \\ \quad -7z = 14 & \rightarrow 3(2) - 4(3) \end{cases}$$

$$\begin{cases} x + y + z = 2 & \\ \quad 4y + 3z = 10 & \\ \quad z = -2 & \rightarrow (3) \div -7 \quad \text{or} \quad -\frac{1}{7} \cdot (3) \end{cases}$$

→ To transform a linear system into an equivalent linear system, we can

1) Interchange 2 equations
2) Multiply an equation by a nonzero constant
3) Add multiply of one equation to another equation and replace the latter equation.

Ex Solve

$$\begin{cases} x - y + 2z = -4 \\ 3x + y - 4z = -6 \\ 2x + 3y - 4z = 4 \end{cases}$$

→ Get rid of x terms in 2nd & 3rd equations then get rid of y term in 3rd equation.

$$\begin{cases} x - y + 2z = -4 & \\ -4y + 10z = -6 & \rightarrow 3(1) - (2) \\ -5y + 8z = -12 & \rightarrow 2(1) - (3) \end{cases}$$

$$\begin{cases} x - y + 2z = -4 & \\ -2y + 5z = -3 & \rightarrow \frac{1}{2} \cdot (2) \\ -5y + 8z = -12 & \end{cases}$$

$$\begin{cases} x - y + 2z = -4 & \\ -2y + 5z = -3 & \\ 9z = 9 & \rightarrow 5(2) - 2(3) \end{cases}$$

$9z = 9 \Rightarrow \boxed{z = 1}$, $-2y + 5(1) = -3 \Rightarrow -2y = -8 \Rightarrow \boxed{y = 4}$

$x - 4 + 2(1) = -4 \Rightarrow x - 2 = -4 \Rightarrow \boxed{x = -2}$

Solution: $(1, 4, -2)$

→ We can get the same types of answers to systems of 3 equations that we got for systems of 2 equations:

1) one solution
2) no solutions ⟶ $0 =$ nonzero number in one row
3) Infinitely many solutions → $0 = 0$ in one row

Ex Physics: An object's vertical position (height) at time t is given by $s = \frac{1}{2}at^2 + V_0 t + S_0$

s = height at time t → feet
a = acceleration from gravity → feet/s^2
V_0 = initial velocity → feet/s
S_0 = initial height → feet

Suppose the height is
$s = 48$ at $t = 1$
$s = 64$ at $t = 2$
$s = 48$ at $t = 3$

find the equation giving the height at time t

$$48 = \frac{1}{2}(a)(1)^2 + V_0(1) + S_0$$
$$64 = \frac{1}{2}(a)(2)^2 + V_0(2) + S_0$$
$$48 = \frac{1}{2}(a)(3)^2 + V_0(3) + S_0$$

$$\begin{cases} \frac{1}{2}a + V_0 + S_0 = 48 \\ 2a + 2V_0 + S_0 = 64 \\ \frac{9}{2}a + 3V_0 + S_0 = 48 \end{cases}$$

$$\begin{cases} a + 2V_0 + 2S_0 = 96 & \to 2\cdot(1) \\ 2a + 2V_0 + S_0 = 64 & \\ 9a + 6V_0 + S_0 = 96 & \to 2\cdot(3) \end{cases}$$

$$\begin{cases} a + 2V_0 + 2S_0 = 96 & \\ 2V_0 + 3S_0 = 128 & \to 2\cdot(1) - (2) \\ 12V_0 + 17S_0 = 768 & \to 9\cdot(1) - (3) \end{cases}$$

$$\begin{cases} a + 2V_0 + 2S_0 = 96 & \\ 2V_0 + 3S_0 = 128 & \\ S_0 = 0 & \to 6\cdot(2) - (3) \end{cases}$$

$\boxed{S_0 = 0}$, $2V_0 + 3(0) = 128 \Rightarrow 2V_0 = 128 \Rightarrow \boxed{V_0 = 64}$

$a + 2(64) + 2(0) = 96 \Rightarrow a + 128 = 96 \Rightarrow \boxed{a = -32}$

So

$$S = \tfrac{1}{2}(-32)t^2 + 64t$$

$$\boxed{S = -16t^2 + 64t}$$

when $t=0$, $S=0$, so object starts out at the ground and initially moves up with speed 64 feet/second, but gravity pulls down at 32 feet/second^2 and object comes back down to the ground.

when? $0 = -16t^2 + 64t \Rightarrow 0 = 16t(-t+4) \Rightarrow t=0$ or $t=4$

Object hits the ground after 4 seconds.

Supplementary Practice Problems: pp. 249-252

3, 5, , 11, 13, 15, 17, 37, 39, 41, 47, 49