Bao Chau Ngo - Kloosterman sheaves for reductive groups
(joint with J. Heinloth and Z. Yun)

Deligne's Kloosterman sheaf

Kloosterman sum of n variables,

let $p$ be prime, $F_q$ a finite extension of $F_p$.

Defn. The Kloosterman sum is

$$K_n(a) = (-1)^{n-1} \sum_{\substack{\sum x_i = a \mod q \cr x_i \in \mathbb{F}_n \cr x_1, \ldots, x_n \in \mathbb{F}_n \cr x_1 x_2 \cdots x_n = q}} \exp \left( \frac{2\pi i}{p} \sum_{i=1}^n x_i \right)$$

Then $|K_n(a)| \leq n^{(n-1)/2}$

For $n = 2$, $\exists \Theta(a) \in [0, \pi)$, $K_2(a) = n^{1/2} \cos \Theta(a)$

equidistributed with respect to the Sato-Tate measure $2 \sin \theta d\theta$

Deligne-Katz solved the equidistribution problem by calculating the geometric monodromy group of the Kloosterman sheaf
Defn of Kloosterman sheaf:

\[ (x_1, \ldots, x_n) \mapsto C_m \quad \text{and} \quad (x_1, \ldots, x_n) \mapsto C_m \quad \text{and} \quad x_1x_n = x_n \]

\[ \sum_{q} \left( \frac{\psi}{q} \right) \]

Given \( \psi: \mathbb{F}_q \to \mathbb{Q} (\mu_p)^* \) a non-trivial character, have Artin-Schreier sheaf \( L_p \) on \( \mathbb{G}_a \)

such that \( \forall a \in \mathbb{F}_q, \quad \text{Tr} (\mathfrak{g}_a, L_p) = \psi (\text{Tr} (a)) \)

Then \( KL_n \approx \Pi, \sigma^* L_p [n-1] \)

\[ KL_n (a) = \text{Tr} (\sigma_{\mathfrak{g}_a}, KL_n) \]

**Theorem (Deligne) \( \Box \):

1. \( KL_n \) is a local system of rank \( g \), pure of weight \( n-1 \).
2. \( KL_n \) is tamely ramified at 0 with regular unipotent monodromy.
(3) $K_{L_n}$ is totally wild at $\infty$, 

$$Sw_\infty(K_{L_n}) = 1$$

Ex: $L_4$ wildly ramified at $\infty$, then

$$Sw_\infty(L_4) = 1$$

$\chi_c(L_4) = 0$, $\chi_c(A') = 1$

Euler characteristic

Have map $\Psi : \Pi_1(G_m, \mathbf{R}) \to \text{GL}_n(\mathbf{Q}_L(\mu_p))$

Then: $G_{\text{geom}} = \text{Un}(q_{\text{geom}}) = \begin{cases} S_p_n & \text{if } n \text{ is even} \\ S_{L_n} & \text{if } n \text{ is odd, } p \text{ odd} \\ S_{D_n} & \text{if } n \text{ is odd, } n \neq 2, p = 2 \\ G_2 & \text{if } n = 7, p = 2 \end{cases}$
Katz asked if any semisimple group can appear using $K_{n}$, and what is the automorphic interpretation?

Gross, Reeder, Frenkel, gave plausible answer for what are the automorphic forms corresponding to such $K_{n}$. Simple supercuspidal representations

Let $G$ be $G(k((s)))$,

$I(0) = \Gamma$, in up

$I(1) = \text{unipotent radical in } I(0)$

$I(2) = [I(1), I(1)]$.

\[ G \subseteq G_{0} \]

Given

\[ \varphi : I(1) \rightarrow G_{0}(k_{p})^{\times} \text{, set } \]

simple supercuspidal C-Field $y$

$I(1)$

let $F$ be global field. Using trace formula, Gross calculated the multiplicity

\[ \mu = \bigotimes \mu_{v} \text{ where } \]

\[ \mu_{v} \text{ is unit-root at } v \in S, \]

and $\mu_{v}$ is either Steinberg or simple supercuspidal at $v$. \]
remaining places

\[ \text{Ex: If } F = k(1) \text{ projective line,} \]

\[ S = \mathcal{O}, \infty \text{ when } \mathcal{O} \text{ is standard} \]

and \( \mathcal{O} \) is simple supercuspidal. Then

\[ m(\Pi) := \text{multiplicity of } \Pi = 1 \]

This \( \Rightarrow \) companion of Kloosterman sheaf

(1) \( K_{\mathcal{L}^+} \) is \( G \)-local system on

\[ \mathbb{P}^1 - \{ \mathcal{O}, \infty \} \]

dual group

(a) \( K_{\mathcal{L}^+} \) is tamely ramified at 0 with

\( G \) regular unipotent monodromy.

(b) let \( K_{\mathcal{L}^+} \) be the local system attached

\[ \text{to the adjoint representation of } \hat{G}. \]

Then \( K_{\mathcal{L}^+} \) is \( S = \sum \chi \text{ of } \hat{G} \) rank of \( \hat{G} \).

Gross - Frenkel construct \( \mathcal{O} \)-modules over \( \mathbb{P}^1 \)

satisfying similar conditions.
1st observation: One can write down explicitly the automorphic forms in the Grass representations.

\[ G(F) < G(\mathfrak{A}) \rightarrow \prod G(\mathfrak{a}) I_0(0) I_\infty(2) = H \]

\[ \lambda \notin \mathfrak{s}_0, \mathfrak{a}_0 \]

\[ \Rightarrow \text{smooth group scheme} \]

\[ \mathcal{Y}_{(0,2)} / P \]

Then

\[ \text{Bun}_{\mathfrak{g}(2)}(h) = \frac{G(\mathfrak{A})}{H} \]

Let \( \tilde{V} = \bigotimes_{v} \tilde{\Pi}_v \) where \( \tilde{\Pi}_v \) is unramified if

\[ v \notin \mathfrak{S}_0, \mathfrak{a}_0 \]

and \( \tilde{\Pi}_v \) is Steinberg

and \( \tilde{\Pi}_0 \) is simple supercuspidal
Assume $G$ is simply connected. Then

$\exists! f \in \Pi_1$

$f : \text{Bun}_n (G) \to C$

$\mathcal{D}_0, 2$

$s.t. f(g, k) = f(g) \varphi(k) \quad \forall k \in \mathcal{I}_0 (1)\)

This determines $f$ uniquely up to a scalar.

We have

$$
\begin{array}{c}
\text{I}_0 (1) \\
\text{I}_2 (1)
\end{array}
\begin{array}{c}
\longrightarrow \\
\downarrow \phi
\end{array}
\begin{array}{c}
\text{Bun}_n \\
\mathcal{D}_1, 2
\end{array}
$$

affine open embeddings

open cell

$f$ is zero on the complement.
$f$ is automatically an eigenvector of the Hecke operator.

Second observation (i.e., (a) above): After upgrading from automorphic form

$\Rightarrow$ eigensheat

$\Rightarrow$ the Kloosterman local system
Let $G$ be a group scheme.

$$G = \{ (\mathfrak{g}, \mathfrak{g}, \mathfrak{g}, \mathfrak{g}) \mid \text{where } x \in X \}$$

$\epsilon, \epsilon' \in \text{Bun}_G$, $\phi : 2 \circ \epsilon / \sim \epsilon' / \phi$.

Let $x \in G_m$, $\mathfrak{g} \in \text{Bun}_G$.

$$G(\mathfrak{g}, x) = \text{affine Grassmanian of } G, G(\mathfrak{a}(\mathfrak{g}))$$

affine Grassmanian of $G$ over $x$

Schubert cells in the affine Grassmanian

indexed by $\lambda \in X^+ = \text{dominant coweights}$
For every dominant coweight, i.e. for every fixed rep V of $\hat{G}$, we have

$$\mathcal{Y}_r V = \mathcal{Y}_r \mathcal{F}$$

$$\mathbb{P} \to \mathbb{P} \times \mathcal{G}$$

$$\text{Eigensheaf: } \left( \mathcal{F} \right) \mathcal{F}_r (\mathcal{F} \mathcal{A} \otimes \mathcal{I} \mathcal{C}(\mathcal{G}^r))$$

Recall $\mathcal{I}(1) / \mathcal{I}(2) \to \mathbb{P} / \mathcal{Y}(0,2)$

Let $A = \mathcal{I} \mathcal{L}_p$. Then $A = j_* \mathcal{L}_p$, and it is strictly eigen.

$$\left( \mathcal{F} \mathcal{A} \right) \mathcal{F}_r (\mathcal{F} \mathcal{A} \otimes \mathcal{I} \mathcal{C}(\mathcal{G}^r)) = A \otimes \mathcal{K} \mathcal{L}^{\dot{V}}$$

Local system attached to $\mathcal{K} \mathcal{L}^{\dot{V}}$ and the representation $V$ of $\hat{G}$.
Lemma: 1) \( \forall V, \mathcal{K}_V^G \) is a perverse sheaf on \( \mathbb{C}_m \).

2) \( \mathcal{K}_V^G \) is a local system.

3) Have a \( G \)-functor

\[
\text{Rep}(G) \rightarrow \text{Loc}(\mathbb{C}_m)
\]

\( V \rightarrow \mathcal{K}_V^G \)

Theorem: 1) \( \mathcal{K}_G \) is tamely ramified at 0 with regular unipotent monodromy \( \mathcal{K}_G \) miniscule.

2) \( \chi_c(\mathbb{C}_m, \mathcal{K}_G^\gamma) = \# \lambda_{\mathfrak{g}} \text{ simple roots of } G \)

\( \chi_c(\mathbb{C}_m, \mathcal{K}_G^{\text{Ad}}(\gamma)) = \text{rank}(G) \).