BRIDGELAND STABILITY CONDITIONS ON THREEFOLDS II: AN APPLICATION TO FUJITA’S CONJECTURE

AREND BAYER, AARON BERTRAM, EMANUELE MACRÌ, AND YUKINOBU TODA

ABSTRACT. We apply a conjectured inequality on third chern classes of stable two-term complexes on threefolds to Fujita’s conjecture. More precisely, the inequality is shown to imply a Reider-type theorem in dimension three which in turn implies that $K_X + 6L$ is very ample when $L$ is ample, and that $5L$ is very ample when $K_X$ is trivial.

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1. INTRODUCTION

A Bogomolov-Gieseker-type inequality on Chern classes of “tilt-stable” objects in the derived category of a threefold was conjectured in [BMT11] in the context of constructing Bridgeland stability conditions. In this paper, we show how the same inequality would allow one to extend Reider’s stable-vector bundle technique ([Rei88]) from surfaces to threefolds, and in particular to obtain Fujita’s conjecture in the threefold case. This follows a line of reasoning that was suggested in [AB09].

While we use the setup of tilt-stability from [BMT11], this paper is intended to be self-contained, and to be readable by birational geometers with a passing familiarity with derived categories.

Tilt-stability depends on two numerical parameters: an ample class $\omega \in \text{NS}_\mathbb{Q}(X)$ and an arbitrary class $B \in \text{NS}_\mathbb{Q}(X)$. It is a notion of stability on a particular abelian category, $\mathcal{B}_{\omega,B}$, of two-term complexes in $D^b(X)$, and codimension three Chern classes of stable

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objects $E$ in this category (and not stable vector bundles) are conjectured to satisfy a Bogomolov-Gieseker inequality in Conjecture 2.3. Assuming this conjecture, we prove the following Reider-type theorem for threefolds:

**Theorem 4.1.** Let $X$ be a smooth projective threefold over $\mathbb{C}$, and let $L$ be an ample line bundle on $X$ such that Conjecture 2.3 holds when $B$ and $\omega$ are scalar multiples of $L$. Fix a positive integer $\alpha$, and assume that $L$ satisfies the following conditions:

1. $L^3 > 49\alpha$;
2. $L^2 . D \geq 7\alpha$, for all integral divisor classes $D$ with $L^2 . D > 0$ and $L . D^2 < \alpha$;
3. $L . C \geq 3\alpha$, for all curves $C$.

Then $H^1(X, K_X \otimes L \otimes I_Z) = 0$ for any zero-dimensional subscheme $Z \subset X$ of length $\alpha$.

Theorem 4.1 would give an effective numerical criterion for an adjoint line bundle to be globally generated ($\alpha = 1$) or very ample ($\alpha = 2$):

**Corollary 1.1** (Fujita’s Conjecture). Let $L$ be an ample line bundle on a smooth projective threefold $X$. Assume Conjecture 2.3 holds for $\omega$ and $B$ as above. Then:

1. $K_X \otimes L^m$ is globally generated for $m \geq 4$. Moreover, if $L^3 \geq 2$, then $K_X \otimes L^3$ is also globally generated.
2. $K_X \otimes L^m$ is very ample for $m \geq 6$.

In Proposition 4.2, we also show (assuming the conjecture) that $K_X \otimes L^5$ is very ample as long as its restriction to special degree one curves is very ample. As a consequence, $K_X \otimes L^5$ is very ample when $K_X$ is trivial, or, more generally, when $K_X . C$ is even for all curves $C \subset X$.

Ein and Lazarsfeld proved that $K_X \otimes L^4$ is globally generated [EL93]. In the case $L^3 \geq 2$, Fujita, Kawamata, and Helmke proved that $K_X \otimes L^3$ is globally generated as well [Fuj93, Kaw97, Hel97]. In fact, in Proposition 4.4, we show that these results conversely give some evidence for Conjecture 2.3. Case (b) in Corollary 1.1 instead is not known in general; but also note that the strongest form of Fujita’s conjecture predicts that $K_X \otimes L^5$ is already very ample. For further references, we refer to [Laz04, Section 10.4]. Notice that the bounds in Theorem 4.1 are very similar to those in [Fuj93] when $\alpha = 1$ (see also [Kaw97, Hel97]) and, when $\alpha = 2$ and $Z$ consists of two distinct points, to those in [Fuj94].

**Approach.** We explain our approach, which was outlined in [AB09, Section 5], but can now be made precise using the strong Bogomolov-Gieseker conjecture of [BMT11]. It generalizes Reider’s original approach [Rei88] by extending it to derived categories.

Suppose the conclusion of Theorem 4.1 is false. Then by Serre duality,

$0 \neq \text{Ext}^2(L \otimes I_Z, \mathcal{O}_X) = \text{Ext}^1(L \otimes I_X, \mathcal{O}_X[1]).$
For appropriate choices of $\omega$ and $B$, both $L \otimes I_X$ and $O_X[1]$ are objects in the abelian category $B_{\omega,B}$, and thus this extension class corresponds to another object $E$ of $B_{\omega,B}$. In Section 3.1, we will show that for $\omega \to 0$, the complex $E$ violates the inequality of Conjecture 2.3, thus it must become unstable. We show in Section 3.2 that the Chern classes of a destabilizing subobject give a contradiction to Assumptions (A) and (B) of the Theorem unless it is of the form $L \otimes I_C$, where $I_C$ is the ideal sheaf of a curve containing $Z$. In Section 4, we apply our conjecture and Assumption (C) to this remaining case and deduce Theorem 4.1.

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**Notation and Convention.** Throughout the paper, $X$ will be a smooth projective threefold defined over $\mathbb{C}$ and $D^b(X)$ its bounded derived category of coherent sheaves. Given a line bundle $L$ on $X$, we will denote by $D_L : D^b(X) \to D^b(X)$ the following local dualizing functor on its derived category:

$$D_L(\_):= (\_)[1] \otimes L = R\text{Hom}(\_ , L[1]).$$

We identify a line bundle $L$ with its first Chern class $c_1(L)$, and write $K_X$ for the canonical line bundle. While $L^{\otimes m}$ denotes the tensor powers of the line bundle, $L^k$ denotes the intersection product of its first Chern class.

**2. Setup**

In this section, we briefly recall the notion of “tilt-stability” defined in [BMT11, Section 3] and its most important properties.

Let $X$ be a smooth projective threefold, and let $\omega, B \in NS_{\mathbb{Q}}(X)$ be rational numerical divisor classes such that $\omega$ is ample. We use $\omega, B$ to define a slope function $\mu_{\omega,B}$ for coherent sheaves on $X$ as follows: For torsion sheaves $E$, we set $\mu_{\omega,B}(E) = +\infty$, otherwise

$$\mu_{\omega,B}(E) = \frac{\omega^2 \tilde{c}_1(E)}{\omega^3 \tilde{c}_0(E)} = \frac{\omega^2 c_1(E)}{\omega^3 c_0(E)} - \frac{\omega^2 B}{\omega^3}$$

where $\tilde{c}_1(E) = e^{-B} c_1(E)$ denotes the Chern character twisted by $B$ (explicitly, $\tilde{c}_0 = \text{rk}$, $\tilde{c}_1 = c_1 - B \text{rk}$, etc.).
A coherent sheaf \( E \) is slope-(semi)stable (or \( \mu_{\omega,B} \)-(semi)stable) if, for all subsheaves \( F \hookrightarrow E \), we have \( \mu_{\omega,B}(F) < (\leq) \mu_{\omega,B}(E/F) \).

Due to the existence of Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to slope-stability, there exists a “torsion pair” \((T_{\omega,B}, F_{\omega,B})\) defined as follows:

\[
T_{\omega,B} = \{ E \in \text{Coh} X : \text{any quotient } E \rightarrow G \text{ satisfies } \mu_{\omega,B}(G) > 0 \}
\]

\[
F_{\omega,B} = \{ E \in \text{Coh} X : \text{any subsheaf } F \hookrightarrow E \text{ satisfies } \mu_{\omega,B}(F) \leq 0 \}
\]

Equivalently, \( T_{\omega,B} \) and \( F_{\omega,B} \) are the extension-closed subcategories of \( \text{Coh} X \) generated by slope-stable sheaves of positive or non-positive slope, respectively.

**Definition 2.1.** We let \( B_{\omega,B} \subset D^b(X) \) be the extension-closure

\[
B_{\omega,B} = \langle T_{\omega,B}, F_{\omega,B} \rangle[1].
\]

More explicitly, \( B_{\omega,B} \) is the subcategory of two-term complexes \( E : E^{-1} \xrightarrow{d} E^0 \) with \( H^{-1}(E) = \ker d \in F_{\omega,B} \) and \( H^0(E) = \cok d \in T_{\omega,B} \). We can characterize isomorphism classes of objects in \( B_{\omega,B} \) by extension classes: to give an object \( E \in B_{\omega,B} \) is equivalent to giving \( T \in T_{\omega,B}, F \in F_{\omega,B} \), and a class \( \xi \in \text{Ext}^2_X(T,F) \).

By the general theory of torsion pairs and tilting [HRS96], \( B_{\omega,B} \) is the heart of a bounded t-structure on \( D^b(X) \). For the most part, we only need that \( B_{\omega,B} \) is an abelian category: exact sequences in \( B_{\omega,B} \) are given by exact triangles in \( D^b(X) \). For any such exact sequence

\[
0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0
\]

in \( B_{\omega,B} \), we have a long exact sequence in \( \text{Coh} X \):

\[
0 \rightarrow H^{-1}(E) \rightarrow H^{-1}(F) \rightarrow H^{-1}(G) \rightarrow
\]

\[
\rightarrow H^0(E) \rightarrow H^0(F) \rightarrow H^0(G) \rightarrow 0.
\]

Using the classical Bogomolov-Gieseker inequality and Hodge Index theorem, we defined the following slope function on \( B_{\omega,B} \): We set \( \nu_{\omega,B}(E) = +\infty \) when \( \omega^2 \tilde{\chi}_1(E) = 0 \), and otherwise

\[
(1) \quad \nu_{\omega,B}(E) = \frac{\omega \tilde{\chi}_2(E) - \frac{1}{6} \omega^3 \tilde{\chi}_0(E)}{\omega^2 \tilde{\chi}_1(E)}.
\]

We showed that this is a slope function, in the sense that it satisfies the weak see-saw property for short exact sequences in \( B_{\omega,B} \): for any subobject \( F \hookrightarrow E \), we have \( \nu_{\omega,B}(F) \leq \nu_{\omega,B}(E/F) \leq \nu_{\omega,B}(E/F) \) or \( \nu_{\omega,B}(F) \geq \nu_{\omega,B}(E) \geq \nu_{\omega,B}(E/F) \).

**Definition 2.2.** An object \( E \in B_{\omega,B} \) is “tilt-(semi)stable” if, for all non-trivial subobjects \( F \hookrightarrow E \), we have

\[
\nu_{\omega,B}(F) < (\leq) \nu_{\omega,B}(E/F).
\]
Motivated by the case of torsion sheaves, by projectively flat vector bundles, and the case of $X = \mathbb{P}^3$, we stated the following conjecture:

**Conjecture 2.3** ([BMT11, Conjecture 1.3.1]). *For any $\nu_{\omega,B}$-semistable object $E \in \mathcal{B}_{\omega,B}$ satisfying $\nu_{\omega,B}(E) = 0$, we have the following inequality*

\[ \tilde{c}_3(E) \leq \frac{\omega^2}{18} \tilde{c}_1(E). \]

Conjecture 2.3 is analogous to the classical Bogomolov-Gieseker inequality, which can be formulated as follows: For any $\mu_{\omega,B}$-semistable sheaf $E$ satisfying $\mu_{\omega,B}(E) = 0$, we have $\omega \tilde{c}_2(E) \leq 0$.

The original motivation for Conjecture 2.3 is to construct examples of Bridgeland stability conditions on $\text{D}^b(X)$. While any linear inequality of the form (2) would be sufficient to this end, the constant $\frac{1}{18}$ in equation (2) is chosen so that, if $\omega$ and $B$ are proportional to the first Chern class of an ample line bundle $L$, the inequality is an equality for tensor power $L^\otimes n$ of $L$. More generally, it is an equality when $E$ is a slope-stable vector bundles $E$ whose discriminant $\Delta = \tilde{c}_1^2 - 2\tilde{c}_0\tilde{c}_2$ satisfies $\omega \Delta(E) = 0$, and for which $\tilde{c}_1(E)$ is proportional to $L$. Such vector bundles have a projectively flat connection, and are examples of tilt-stable objects:

**Proposition 2.4** ([BMT11, Proposition 7.4.1]). *Let $L$ be an ample line bundle, and assume that both $\omega$ and $B$ are proportional to $L$. Then any slope-stable vector bundle $E$, with $\omega \Delta(E) = 0$ and for which $\tilde{c}_1(E)$ is proportional to $L$, is also tilt-stable with respect to $\nu_{\omega,B}$.*

The proof is essentially the same as for line bundles $L^\otimes n$ in [AB09, Proposition 3.6].

By assuming Conjecture 2.3, we can also show conversely: if an object in $\mathcal{B}_{\omega,B}$ is tilt-stable and the inequality in Conjecture 2.3 is an equality, then it must have trivial discriminant. We first recall that, based on Bridgeland’s deformation theorem in [Bri07], we also showed the existence of a continuous family of stability conditions depending on real classes $\omega, B$:

**Proposition 2.5** ([BMT11, Corollary 3.3.3]). *Let $U \subset \text{NS}_\mathbb{R}(X) \times \text{NS}_\mathbb{R}(X)$ be the subset of pairs of real classes $(\omega, B)$ for which $\omega$ is ample. There exists a notion of “tilt-stability” for every $(\omega, B) \in U$. For every object $E$, the set of $(\omega, B)$ for which $E$ is $\nu_{\omega,B}$-stable defines an open subset of $U$.***

By using Proposition 2.5, we can then prove the following.

**Proposition 2.6.** *Let $L$ be an ample line bundle, and assume that both $\omega$ and $B$ are proportional to $L$. Assume also that Conjecture 2.3 holds for such $B$ and $\omega$. Let $E \in \mathcal{B}_{\omega,B}$ be*
a $\nu_{\omega,B}$-stable object, with $\check{c}h_0(E) \neq 0$ and $\check{c}h_1(E)$ proportional to $L$, satisfying:

$$\frac{\omega^3}{6} \check{c}h_0(E) = \omega \check{c}h_2(E) \quad \text{and} \quad \check{c}h_3(E) = \frac{\omega^2}{18} \check{c}h_1(E).$$

Then $\omega \Delta(E) = 0$.

**Proof.** Write $d = L^3$, $B = b_0 L$, $\omega = T_0 L$, $\check{c}h_0(E) = r$, and $\check{c}h_1(E) = c L$. The idea for the proof is that, since stability is an open property, we can deform $b = b_0$ and $T = T(b)$ slightly such that $E$ is still $\nu_{T,L,bL}$-stable with $\nu_{T,L,bL}(E) = 0$, but violates Conjecture 2.3.

Evidently, $\nu_{T,L,bL}(E) = 0$ is equivalent to

$$T^2 = \frac{6}{rd} \left( L \cdot \check{c}h_2(E) - b^2 cd + \frac{b^2}{2} rd \right).$$

Since $T_0 > 0$, and since the equation is satisfied for $T = T_0$ and $b = b_0$, the equation defines a function $T = T(b)$ for $b$ nearby $b_0$. Consider the function

$$f = f(b) = \check{c}h_3(E) - \frac{\omega^2}{18} \check{c}h_1(E)$$

$$= \check{c}h_3(E) - b L \cdot \check{c}h_2(E) + \frac{b^2}{2} cd - \frac{b^3}{6} rd - \frac{1}{18} T(b)^2 (c - rb) d.$$

By Proposition 2.5 and Conjecture 2.3, we have $f(b) \leq 0$ for $b$ close to $b_0$, and $f(b_0) = 0$. Hence

$$0 = f'(b_0) = \frac{1}{3r} L \cdot \Delta(E).$$

Finally, based on an alternate construction of tilt-stability, we also showed that it behaves well with respect to the dualizing functor $D_L(\_)= R\text{Hom}(\_, L[1])$ for every line bundle $L$. For this purpose, we fix $B = \frac{L}{2}$:

**Proposition 2.7.** Let $F \in B_{\omega, \frac{L}{2}}$ be an object with $\nu_{\omega,B}(A) < +\infty$ for every subobject $A \subset F$. Then there is an exact triangle $\tilde{F} \rightarrow D_L(F) \rightarrow T_0[-1]$ where $T_0$ is a zero-dimensional torsion sheaf and $\tilde{F}$ an object of $B_{\omega, \frac{L}{2}}$ with $\nu_{\omega, \frac{L}{2}}(\tilde{F}) = -\nu_{\omega, \frac{L}{2}}(F)$. The object $\tilde{F}$ is $\nu_{\omega, \frac{L}{2}}$-semistable if and only if $F$ is $\nu_{\omega, \frac{L}{2}}$-semistable.

**Proof.** Since $D_L(\_)$ can be written as the composition $\_ \otimes L \circ D(\_)$, this follows from [BMT11, Proposition 5.1.3] and the fact that tensoring with $L$ corresponds to replacing $B$ with $B+L$.

$\square$
3. Reduction to curves

In this section, we use Assumptions (A) and (B) of Theorem 4.1 to show that the non-vanishing of \( H^1(X, K_X \otimes L \otimes I_Z) \) implies the existence of special low-degree curves on \( X \). The approach, explained in the introduction, involves studying the tilt-stability of a certain object \( E \) in the category \( B \) constructed in the previous section.

3.1. Bogomolov-Gieseker inequalities and stability. We will use Conjecture 2.3 in the case where \( L \) is an ample line bundle on \( X \), \( \omega = TL \) for some \( T > 0 \), and \( B = L^2 \); in other words, \( \tilde{\text{ch}}(\_ \_ \_) = \text{ch}(\_ \_ \_) \cdot e^{-L/2} \). The abelian category \( B := B_{TL, L^2} \) is independent of \( T \).

To simplify notation, we will rescale the slope function: set \( t = T^2/6 \) and write \( \nu_t \) for \( \nu_T \).

\[
\nu_t(\_ \_ \_) = T \cdot \nu_{TL, \frac{L}{T}}(\_ \_ \_) = \frac{L \cdot \tilde{\text{ch}}_2(\_ \_ \_) - td \tilde{\text{ch}}_0(\_ \_ \_)}{L^2 \cdot \tilde{\text{ch}}_1(\_ \_ \_)}, \tag{3}
\]

where \( d := L^3 \). Then the inequality of Conjecture 2.3 states that, for every \( \nu_t \)-stable object \( E \), we have

\[
\tilde{\text{ch}}_3(E) \leq \frac{t}{3} L^2 \cdot \tilde{\text{ch}}_1(E) \quad \text{if} \quad L \cdot \tilde{\text{ch}}_2(E) = dt \tilde{\text{ch}}_0(E). \tag{4}
\]

Let \( Z \subset X \) be a zero-dimensional subscheme of length \( \alpha \). Following [AB09], observe that if \( H^1(X, K_X \otimes L \otimes I_Z) \neq 0 \), then by Serre duality, we also have \( \text{Ext}^2(L \otimes I_Z, \mathcal{O}_X) \neq 0 \). Any non-zero element \( \xi \in \text{Ext}^2(L \otimes I_Z, \mathcal{O}_X) \) gives a non-trivial exact triangle in \( D^b(X) \)

\[
\mathcal{O}_X[1] \to E = E_\xi \to L \otimes I_Z \xrightarrow{\xi} \mathcal{O}_X[2]. \tag{5}
\]

We will show that \( E \) is \( \nu_t \)-semistable for \( t = \frac{1}{8} \); its Chern classes invalidate the inequality of Conjecture 2.3 for \( t \ll 1 \), and thus it must become unstable for \( t < t_0 \) and some \( t_0 \in (0, \frac{1}{8}] \); finally, we will show that the the Chern classes of its destabilizing factor would give special curves or divisors on \( X \).

**Proposition 3.1.** Assume that \( H^1(X, K_X \otimes L \otimes I_Z) \neq 0 \), and let \( E \) be an extension as given by equation (5).

(a) \( E \in B \) and

\[
\tilde{\text{ch}}(E) = \left( 0, L, 0, \frac{d}{24} - \alpha \right).
\]

(b) If \( t > \frac{1}{8} \), then (5) destabilizes \( E \) with respect to \( \nu_t \).

(c) If \( t = \frac{1}{8} \), then \( E \) is \( \nu_t \)-semistable.

(d) Assume Conjecture 2.3 and Assumption (A) of Theorem 4.1. Then \( E \) is not \( \nu_t \)-semistable for \( 0 < t \ll 1 \),
Proof. First of all, we have
\[
\widetilde{\text{ch}}(O_X) = \left(1, -\frac{L}{2}, \frac{L^2}{8}, -\frac{L^3}{48}\right),
\]
\[
\widetilde{\text{ch}}(L \otimes I_Z) = \left(1, \frac{L}{2}, \frac{L^2}{8}, \frac{L^3}{48} - \alpha\right).
\]
As $O_X$ and $L \otimes I_Z$ are slope-stable, with $\mu_{\omega,B}(O_X) < 0$ and $\mu_{\omega,B}(L \otimes I_Z) > 0$, we have $O_X \in \mathcal{F}$ and $L \otimes I_Z \in \mathcal{T}$. By the definition of $B$, it follows that $O_X[1], L \otimes I_Z$ and $E$ are all objects of $B$; in particular, we have proved (a).

Moreover, we have
\[
\nu_t(O_X[1]) = 2 \left(t - \frac{1}{8}\right), \quad \nu_t(E) = 0
\]
which immediately implies (b), since (5) is an exact sequence in $B$.

To prove (c), simply observe that, by Proposition 2.4, both $O_X[1]$ and $L$ are $\nu_t$-stable for all $t > 0$. Moreover, since $\nu_t(L \otimes I_Z) = \nu_t(L)$, any destabilizing subobject $A \hookrightarrow L \otimes I_Z$ would also destabilize $L$ via the composition $A \hookrightarrow L \otimes I_Z \hookrightarrow L$ (which is an inclusion in $B$); thus $L \otimes I_Z$ is also $\nu_t$-stable. For $t = \frac{1}{8}$, we have $\nu_t(O_X[1]) = \nu_t(L \otimes I_Z) = 0$, and thus the extension (5) shows that $E$ is $\nu_t$-semistable at $t = \frac{1}{8}$.

Finally, if $E$ was $\nu_t$-semistable for all $t \in (0, \frac{1}{8}]$, then by our conjectural inequality (4) we would get
\[
\frac{d}{24} - \alpha \leq \frac{t}{3} d
\]
for all such $t$. Hence $d \leq 24\alpha$, in contradiction to Assumption (A).

Notice that the previous proposition would answer Question 4 in [AB09]. Also observe that in part (d), instead of Assumption (A), already assuming $d > 24\alpha$ would have been enough. Similarly, instead of Conjecture 2.3, any linear inequality between $\widetilde{\text{ch}}_3$ and $\widetilde{\text{ch}}_1$ would have been sufficient.

In the following proposition, we will show that our situation is self-dual with respect to the local dualizing functor $D_L(\_): \mathcal{D}_{\text{RHom}(\_ \otimes L[1])}$. As a preliminary, let us first note that we may make the following assumption:

(*) $H^1(X, K_X \otimes L \otimes I_Z') = 0$ for all subschemes $Z' \subsetneq Z$, and $H^1(X, K_X \otimes L \otimes I_Z) \cong \mathbb{C}$.

Indeed, in order to show $H^1(X, L \otimes I_Z \otimes K_X) = 0$, we can proceed by induction on the length of $Z$ (the case $\alpha = 0$ is, of course, given by Kodaira vanishing).

**Proposition 3.2.** If Assumption (*) holds, and $E$ is given by the unique non-trivial extension of the form (5), then $E \cong D_L(E)$.

*Proof.* Due to Assumption (*), it is sufficient to show that $D_L(E)$ is again a non-trivial extension of the form (5). Applying the octahedral axiom to the composition $O_Z[-1] \rightarrow
Given by Conjecture 2.3 implies the existence of 3.2. Chern classes of destabilizing subobjects. 2 of the form (5).

\[
\text{for every inclusion } k(x) \hookrightarrow O_Z. \text{ Given such an inclusion, let } Z' \subset Z \text{ be the subscheme given by } O_{Z'} \cong O_Z/k(x). \text{ If the composition (9) vanishes, then } \xi \text{ factors via } L \otimes I_Z \hookrightarrow L \otimes I_{Z'}. \text{ This contradicts our assumption } \text{Ext}^2(L \otimes I_{Z'}, O_Z) = H^1(X, L \otimes I_{Z'} \otimes K_X)^\vee = 0.
\]

Now we apply $\mathbb{D}_L$ to the exact triangle $O_X[1] \to F \to O_Z[-1]$. As $\mathbb{D}_L(O_X[1]) = L$ and $\mathbb{D}_L(O_Z[-1]) = O_Z[-1]$, dualizing (8) gives an exact triangle $O_Z[-1] \to \mathbb{D}_L(F) \to L \to O_Z$. Since $\text{Hom}(\mathbb{D}_L(F), k(x)[-1]) = \text{Hom}(k(x)[-1], F) = 0$ for all $x \in X$, the map $L \to O_Z$ must be surjective, and hence $\mathbb{D}_L(F) \cong L \otimes I_Z$. Consequently, applying $\mathbb{D}_L$ to the exact triangle $F \to E \to L$ shows that $\mathbb{D}_L(E)$ is indeed a non-trivial extension of the form (5).

3.2. Chern classes of destabilizing subobjects. By Proposition 3.1 and Proposition 2.5, Conjecture 2.3 implies the existence of $t_0 \in (0, \frac{\alpha}{K}]$ with the following properties:

- $E$ is $\nu_{t_0}$-semistable.
- There exists an exact sequence in $B$

\[
0 \to A \to E \to F \to 0,
\]

with $\nu_t(A) > 0$ if $t < t_0$, and $\nu_{t_0}(A) = 0$.

In the remainder of this section, we will prove the following statement:

**Proposition 3.3.** Assume that $X, L, \alpha$ satisfy Assumptions (A) and (B) of Theorem 4.1 and Assumption (*) of the previous section. Then in any destabilizing sequence (10), the object $A$ is of the form $L \otimes I_C$, for some purely one-dimensional subscheme $C \subset X$ containing $Z$.

We will first prove this for subobjects satisfying $L^2.\tilde{c}_1(A) \leq L^2.\tilde{c}_1(F)$, or, equivalently,

\[
L^2.\tilde{c}_1(A) \leq \frac{1}{2}L^2.\tilde{c}_1(E) = \frac{d}{2}.
\]

(We will later use the derived duality $\mathbb{D}_L(\_)$ to reduce to this case.)

**Lemma 3.4.** Any subobject $A$ satisfying (11) is a sheaf with $\text{rk}(A) = \text{rk}(H^0(A)) > 0$.  


Proof. Consider the long exact cohomology sequence for $A \hookrightarrow E \twoheadrightarrow F$. If $H^{-1}(A) \neq 0$, then $H^{-1}(A) = \mathcal{O}_X$ because $H^{-1}(F)$ is torsion-free. Then $H^0(A)$ is also torsion-free, and (11) implies

$$L^2 \cdot \tilde{c}_1(H^0(A)) = L^2 \cdot \tilde{c}_1(A) - L^2 \cdot \tilde{c}_1(\mathcal{O}_X[1]) \leq \frac{d}{2} - \frac{d}{2} = 0.$$ 

On the other hand, by construction of $B$, every HN-filtration factor $U$ of $H^0(A)$ satisfies $L^2 \cdot \tilde{c}_1(U) > 0$; thus $H^0(A) = 0$ and $A = \mathcal{O}_X[1]$. This contradiction proves $H^{-1}(A) = 0$.

Finally, note that if $A = H^0(A)$ is a torsion-sheaf, then $\nu_t(A)$ is independent of $t$, again a contradiction.

Lemma 3.5. Either $A$ is torsion-free, or its torsion-part $A_t$ satisfies

$$L^2 \cdot \text{ch}_1(A_t) - 2L \cdot \text{ch}_2(A_t) \geq 0 \quad \text{and} \quad L^2 \cdot \text{ch}_1(A_t) > 0.$$ 

Proof. The sheaf $A_t$ is a subobject of $E$ in $\mathcal{B}$ with $\text{rk} = 0$. Hence $L \cdot \tilde{c}_2(A_t) \leq 0$, otherwise it would destabilize $E$ at $t = \frac{1}{8}$. Expanding $\tilde{c}_2$ gives the first inequality. To show the second inequality, we just observe that there are no non-trivial morphisms from sheaves supported in dimension $\leq 1$ to $E$.

Lemma 3.6. In the HN-filtration of $A$ with respect to slope-stability, there exists a factor $U$ of rank $r$ such that $\Gamma := L - \frac{\text{ch}_1(U)}{r}$ satisfies the following inequalities:

(I) \hspace{2cm} L^2 \cdot \Gamma \leq L \cdot \Gamma^2 + 6 \alpha \\
(II) \hspace{2cm} \frac{d}{2} \left(1 - \frac{1}{r}\right) \leq L^2 \cdot \Gamma < \frac{d}{2}.

The case $r = 1$ and $L^2 \cdot \Gamma = 0$ only occurs when $A$ is a torsion-free sheaf of rank one and $H^{-1}(F) = \mathcal{O}_X$.

If $A$ was a line bundle, the above definition of $\Gamma$ would be just as Reider’s original argument for surfaces: in this case, $\Gamma$ is the support of the cokernel of $A \hookrightarrow H^0(E) \cong L \otimes I_Z$.

Proof. From $\nu_{t_0}(A) = 0$ we obtain

(12) \hspace{2cm} t_0 = \frac{L \cdot \tilde{c}_2(A)}{\text{rk}(A)d}.

Applying the conjectured inequality (4) to $E$, and plugging in $t_0$ gives

$$\frac{d}{24} - \alpha = \tilde{c}_3(E) \leq \frac{L^2 \cdot \tilde{c}_1(E)}{3} t_0 = \frac{d L \cdot \tilde{c}_2(A)}{3 \text{rk}(A)d} = \frac{1}{3} \frac{L \cdot \tilde{c}_2(A)}{\text{rk}(A)}.$$
We want to bound $L. \tilde{\text{ch}}_2(A)$. First we expand $\tilde{\text{ch}}_2(A)$:

$$\tilde{\text{ch}}_2(A) = \text{ch}_2(A) - \frac{L. \text{ch}_1(A)}{2} + \text{rk}(A)\frac{L^2}{8}.$$ 

Substituting, we deduce

$$L^2 \frac{\text{ch}_1(A)}{\text{rk}(A)} - 2 L. \frac{\text{ch}_2(A)}{\text{rk}(A)} \leq 6 \alpha. \tag{13}$$

Let $A_{tf}$ denote the torsion-free part of $A$, and consider its HN-filtration. Among the HN factors, we choose a torsion-free sheaf $U$ for which the function

$$\eta(\_):= L^2 \frac{\text{ch}_1(\_)}{\text{rk}(\_)} - 2 L. \frac{\text{ch}_2(\_)}{\text{rk}(\_)}$$

is minimal. Notice that $\eta$ satisfies the see-saw property: for an exact sequence of torsion-free sheaves

$$0 \to M \to N \to P \to 0,$$

we have $\eta(N) \geq \min\{\eta(M), \eta(P)\}$. Hence we get a chain of inequalities leading to

$$\eta(U) \leq \eta(A_{tf}) \leq \eta(A) \leq 6 \alpha \tag{14}$$

where we used Lemma 3.5 for the second inequality.

To abbreviate, we now write $D := \text{ch}_1(U)$ and $r := \text{rk}(U)$. Since $U$ is $\mu_L$-semistable, we can combine the classical Bogomolov-Gieseker inequality with (14) to obtain

$$L^2 \frac{D}{r} = \frac{2 L. \text{ch}_2(U)}{r} + \eta(U) \leq L. \frac{D^2}{r^2} + 6 \alpha.$$

Substituting $D = rL - r\Gamma$ yields the inequality (I).

To prove the chain of inequalities (II), we observe on the one hand that $L^2. \tilde{\text{ch}}_1(U) > 0$ by the definition of $T_{\omega,B} = B \cap \text{Coh} X$. On the other hand, $U$ is a subquotient of $A$ in $T_{\omega,B}$; combined with inequality (11) we obtain

$$0 < L^2. \tilde{\text{ch}}_1(U) \leq L^2. \tilde{\text{ch}}_1(A) \leq d \frac{d}{2}.$$

Plugging in $\tilde{\text{ch}}_1(U) = -\frac{\tau}{2}L + D = \frac{\tau}{2}L - r\Gamma$ shows the inequality (II).

Finally, note that in the case $r = 1$ and $L^2. \Gamma = 0$ the chain of inequalities leading to the first part of (II) must be equalities; in particular $L^2. \tilde{\text{ch}}_1(U) = L^2. \tilde{\text{ch}}_1(A)$. This shows that $A_{tf}$ cannot have any other HN-filtration factors besides $U$, i.e., $U = A_{tf}$. Additionally it implies that $\tilde{\text{ch}}_1(A_t) = 0$, in contradiction to Lemma 3.5; hence $A_t = 0$ and $A = U$ is a torsion-free rank one sheaf.

As $L \otimes I_Z$ is torsion-free, if the image of $H^{-1}(F) \to A$ is non-trivial, then the map is surjective, and the inclusion $A \hookrightarrow E$ factors via $A \hookrightarrow O_X[1] \hookrightarrow E$, in contradiction to the stability of $O_X[1]$ for all $t$ and $\nu_{t_0-\varepsilon}(A) > 0 > \nu_{t_0-\varepsilon}(O_X[1])$. Thus $H^{-1}(F) = O_X$. 

$\square$
Proof. (Proposition 3.3) We combine (I) and (II) with the Hodge Index Theorem (just as in [AB09, Corollary 3.9]) to obtain
\[
(L.\Gamma^2) d \leq \frac{d}{2} (L.\Gamma^2 + 6\alpha),
\]
and so \(L.\Gamma^2 \leq 6\alpha\).

In the case \(r > 1\), we use (I) and (II) again to get
\[
d^4 \leq L.\Gamma \leq L.\Gamma^2 + 6\alpha \leq 12\alpha,
\]
and so \(d \leq 48\alpha\) in contradiction to Assumption (A).

Reider’s original argument in [Rei88] deals with the case \(r = 1\): In case \(L.\Gamma \neq 0\), then \(L.\Gamma \geq 1\). Let \(\kappa := L.\Gamma^2 \leq 6\alpha\). Again combining the Hodge Index Theorem with (I), we obtain
\[
(L.\Gamma^2) d \leq (L.\Gamma^2 + 6\alpha)^2,
\]
and so
\[
d \leq 12\alpha + \frac{\kappa^2 + 36\alpha^2}{\kappa}.
\]
The RHS is strictly decreasing function for \(\kappa \in (0, 6\alpha]\) and equals \(49\alpha\) for \(\kappa = \alpha\); thus Assumption (A) implies \(\kappa < \alpha\). On the other hand, \(\Gamma\) is integral, and hence Assumption (B) implies \(L.\Gamma \geq 7\alpha\), in contradiction to (I).

Finally, if \(L.\Gamma = 0\); then, according to Lemma 3.6, we have \(H^{-1}(F) \cong \mathcal{O}_X\). Hence \(A\) is a subsheaf of \(L \otimes I_Z\) with \(\tilde{\text{ch}}_1(A) = \text{ch}_1(L)\); this is only possible if \(A \cong L \otimes I_W\), for some closed subscheme \(W \subset X\) with \(\dim(W) \leq 1\). If \(W\) is zero-dimensional, then \(\tilde{\text{ch}}_2(A) = \frac{1}{2}L^2\) and equation (12) gives \(t_0 = \frac{1}{2}\), in contradiction to \(t_0 \in (0, \frac{1}{8}]\). Hence \(W\) is one-dimensional, and we have shown that any subobject \(A\) with \(\tilde{\text{ch}}_1(A) \leq \frac{d}{2}\) is of the form \(A \cong L \otimes I_W\). In particular \(\tilde{\text{ch}}_1(A) = \frac{d}{2}\) in this case, so there are no subobject with \(\tilde{\text{ch}}_1(A) < \frac{d}{2}\).

Now assume \(\tilde{\text{ch}}_1(A) > \frac{d}{2}\). We can apply Proposition 3.2 and Proposition 2.7 to the short exact sequence (10) obtain a short exact sequence in \(\mathcal{B}\)
\[
0 \to \tilde{F} \xrightarrow{u} E \to E/\tilde{F} \to 0
\]
which is again destabilizing. Indeed, since \(\mathcal{B}\) is the heart of a bounded t-structure, there exists a cohomology functor \(H^*_B(\_\_\_\_)\). Applied to the exact triangle
\[
\mathbb{D}_L(F) \to \mathbb{D}_L(E) = E \to \mathbb{D}_L(A),
\]
it induces a long exact sequence in \(\mathcal{B}\)
\[
0 \to \tilde{F} = H^0_B(\mathbb{D}_L(F)) \xrightarrow{u} E \to \tilde{A} \to T_0 = H^1_B(\mathbb{D}_L(F)) \to 0.
\]
As $\mathbb{D}_L$ preserves $L^{2}, \tilde{c}_1(\_)$, we have that $\tilde{F}$ is a destabilizing subobject with $\tilde{c}_1(F) = \tilde{c}_1(E) - \tilde{c}_1(A) < \frac{d}{2}$, which does not exist.

Finally, note that the long exact sequence (15) also implies that $\mathbb{D}_L(A) = \tilde{A} \in \mathcal{B}$. This gives the vanishing of $0 = \operatorname{Hom}(\mathbb{D}_L(A), k(x)[-1]) = \operatorname{Hom}(k(x)[-1], A)$. This is equivalent to the claim that $W$ is a purely one-dimensional scheme, as any subsheaf $k(x) \hookrightarrow \mathcal{O}_W$ gives an extension of $k(x)$ by $L \otimes I_W$. This finishes the proof of Proposition 3.3.

\[\Box\]

4. A REIDER-TYPE THEOREM

In this section we prove our main theorem:

**Theorem 4.1.** Let $L$ be an ample line bundle on a smooth projective threefold $X$, and assume Conjecture 2.3 holds for $B$ and $\omega$ proportional to $L$. Fix a positive integer $\alpha$, and assume that $L$ satisfies the following conditions:

\begin{enumerate}
\item[(A)] $L^3 > 49\alpha$;
\item[(B)] $L^2.D \geq 7\alpha$, for all integral divisor classes $D$ with $L^2.D > 0$ and $L.D^2 < \alpha$;
\item[(C)] $L.C \geq 3\alpha$, for all curves $C$.
\end{enumerate}

Then $H^1(X, K_X \otimes L \otimes I_Z) = 0$, for any zero-dimensional subscheme $Z \subset X$ of length $\alpha$.

**Proof.** As explained in Section 3.1, we may proceed by induction on the length of $Z$ and may use Assumption (*). Let $t_0 \in (0, \frac{1}{8}]$ be as in Section 3.2 and let $t = t_0 - \epsilon$. Truncating the Harder-Narasimhan filtration of $E$ with respect to $\nu_t$-stability gives a short exact sequence

$$0 \rightarrow A \rightarrow E \rightarrow F \rightarrow 0$$

with $\nu_t(A) > 0$, such that any subobject $A' \hookrightarrow E$ with $\nu_t(A') > 0$ factors via $A' \hookrightarrow A$. By Proposition 3.3, $A$ is of the form $L \otimes I_C$ for some purely one-dimensional subscheme $C \subset X$; it also implies that $A$ is stable, as any destabilizing subobject $A'$ of $A$ would again be of the form $A' \cong L \otimes I_{C'}$, so that the quotient $A/A'$ would be a torsion sheaf with $\nu_t(A/A') = +\infty$.

Let $\tilde{F}$ be the object obtained by dualizing $F$ and applying Proposition 2.7. The map $\mathbb{D}_L(F) \rightarrow \mathbb{D}_L(E) \cong E$ induces a map $\tilde{F} \rightarrow E$ which is an injection in $\mathcal{B}$. Since

\begin{equation}
\tilde{c}_i(\tilde{F}) = \tilde{c}_i(\mathbb{D}_L(F))
\end{equation}

for $i \leq 2$, we have $\nu_t(\tilde{F}) = -\nu_t(F) > 0$; thus the map factorizes as $\tilde{F} \hookrightarrow A \hookrightarrow E$. By Proposition 3.3, the object $\tilde{F}$ is of the form $L \otimes I_{C'}$ for some purely one-dimensional subscheme $C' \subset X$. Equation (16) also implies $\tilde{c}_3(\tilde{F}) = \tilde{c}_3(A)$ for $i \leq 2$; thus the (non-trivial) map $L \otimes I_{C'} \rightarrow L \otimes I_C$ has zero-dimensional cokernel. It follows that

$$\tilde{c}_3(F) = \tilde{c}_3(\mathbb{D}_L(F)) \leq \tilde{c}_3(\tilde{F}) \leq \tilde{c}_3(A).$$
This implies that
\[
2\tilde{c}_3(A) \geq \tilde{c}_3(A) + \tilde{c}_3(F) = \tilde{c}_3(E) = \frac{d}{24} - \alpha,
\]
and the difference of the two sides is a non-negative integer.

On the other hand, as $A$ is stable, by Conjecture 2.3, by (12) and (17), and by expanding $\tilde{c}_h$ we have
\[
(18) \quad \frac{d}{48} - \frac{\alpha}{2} \leq \tilde{c}_3(A) \leq \frac{t_0}{3} L^2 \tilde{c}_1(A) = \frac{1}{6} L. \tilde{c}_2(A) = \frac{d}{48} - \frac{L.C}{6}.
\]

We now use Assumption (C): $L.C \geq 3\alpha$. This contradicts (18), unless $L.C = 3\alpha$ and
\[
\frac{d}{48} - \frac{\alpha}{2} = \tilde{c}_3(A) = \frac{t_0}{3} L^2 \tilde{c}_1(A).
\]

Since $(TL).\Delta(A) = T3\alpha \neq 0$, this in turn contradicts Proposition 2.6. □

We also obtain the following result characterizing the only possible counter-examples to Fujita’s very ampleness conjecture in case $L = M^5$:

**Proposition 4.2.** Assume that Conjecture 2.3 holds for $X$, $\omega = tL$ and $B = \frac{L}{2}$ and $L \cong M^5$ for an ample line bundle $M$. Then either $K_X \otimes L$ is very ample, or there exists a curve $C$ of degree $M.C = 1$ and arithmetic genus $g_a(C) = \frac{5}{2} + \frac{1}{2} K_X.C$ such that $K_X \otimes L|_C$ is a line bundle of degree $2g_a(C)$ on $C$ which is not very ample.

**Proof.** Assume that $K_X \otimes L$ is not very ample. We follow the logic and the notation of the proof of Theorem 4.1, with $\alpha = 2$. As before, let $A = L \otimes I_C$ be the destabilizing subobject of $E$ for $t = t_0 - \epsilon$; here $C$ is a purely one-dimensional subscheme of $X$. By the proof of Theorem 4.1, we have $L.C < 6$ and thus necessarily $M.C = 1$ and $L.C = 5$. In particular, $C$ is reduced and irreducible. We claim that $\tilde{c}_3(A) = \frac{d}{48} - 1$. Indeed, setting $\alpha = 2$ in (18) gives
\[
(19) \quad \frac{d}{48} - 1 \leq \tilde{c}_3(A) \leq \frac{d}{48} - \frac{5}{6}.
\]

On the other hand, if $\tilde{c}_3(A) \neq \frac{d}{48} - 1$, then, by (17), $\tilde{c}_3(A) \geq \frac{d}{48} - \frac{1}{2}$, a contradiction to the inequality (19).

From the claim, we obtain
\[
ch_3(L \otimes O_C) = ch_3(L) - ch_3(A) = \frac{7}{2}
\]
and thus
\[
ch_3(O_C) = ch_3(L \otimes O_C) - L.C = -\frac{3}{2}
\]
By Hirzebruch-Riemann-Roch, we get
\[
1 - g_a(C) = ch_3(O_C) - \frac{1}{2} K_X.C.
\]
Plugging in the previous equation and solving for $K_X.C$ shows that $K_X \otimes L|_C$ is a line bundle of degree $2g_a(C)$ on $C$. The explicit expression for $g_a(C)$ follows immediately.

Finally, the cohomology sheaves of the quotient $F \cong E/A$ are $H^{-1}(F) \cong \mathcal{O}_X$ and $H^0(F) \cong L \otimes \mathcal{O}_C(-Z)$ (where $\mathcal{O}_C(-Z)$ denotes the ideal sheaf of $Z \subset C$). If $F$ were decomposable, $\tilde{F}$ would be a decomposable destabilizing subobject of $E$, which cannot exist. Hence

$$0 \neq \operatorname{Ext}^2(L \otimes \mathcal{O}_C(-Z), \mathcal{O}_X) = H^1(C, K_X \otimes L|_C(-Z))^\vee.$$ 

On the other hand, $K_X \otimes L|_C$ is a line bundle of degree $2g_a(C)$ on an irreducible Cohen-Macaulay curve, and thus $H^1(K_X \otimes L|_C) = 0$. Hence $K_X \otimes L|_C$ is not very ample.

**Remark 4.3.** Notice that Proposition 4.2 implies Fujita’s conjecture when $K_X$ is numerically trivial (or, more generally, when $K_X.C$ is even for all integral curve classes $C$).

In case the curve $C \subset X$ of Proposition 4.2 is l.c.i, one can be even more precise. Let $\omega_C$ be the dualizing sheaf (which agrees with the dualizing complex, as $\mathcal{O}_C$ is pure and thus $C$ Cohen-Macaulay). The sheaf $K_X \otimes L(-Z)|_C$ is torsion-free of rank one and degree $2g_a(C) - 2$ with $H^1(K_X \otimes L(-Z)|_C) \neq 0$, and thus Serre duality implies $K_X \otimes L(-Z)|_C \cong \omega_C$. If $N$ is the normal bundle, adjunction gives $\Lambda^2 N \cong L(-Z)$. In particular, the normal bundle has degree 3. Since $M.C = 1$, bend-and-break implies that such a curve cannot be rational.

In conclusion, we show how to reverse part of the argument in this section when $Z$ has length one. Indeed, in such a case we can use Ein-Lazarsfeld theorem (or better, its variant by Kawamata and Helmke) to show that Conjecture 2.3 holds true for this particular case, coherently with our result:

**Proposition 4.4.** Let $L$ be an ample line bundle on a smooth projective threefold $X$. Assume that $L$ satisfies the following conditions:

(a) $L^3 \geq 28$;
(b) $L^2.D \geq 9$, for all integral effective divisor classes $D$.

Assume also that there exists $x \in X$ such that $H^1(X, K_X \otimes L \otimes I_x) \neq 0$. Then Conjecture 2.3 holds for all objects $E \in \mathcal{B}$ given as non-trivial extensions

$$\mathcal{O}_X[1] \to E \to L \otimes I_x \to \mathcal{O}_X[2].$$

**Proof.** The argument is very similar to [Kaw97], Proposition 2.7 and Theorem 3.1, Step 1. We freely use the notation from [Laz04, Sections 9 & 10]. By [Kaw97, Lemma 2.1], given a rational number $t$ satisfying $3/\sqrt[3]{t^3} < t < 1$, there exists a $\mathbb{Q}$-divisor $D$ numerically equivalent to $tL$ such that $\operatorname{ord}_x D = 3$. Let $c \leq 1$ the log-canonical threshold of $D$ at $x$. 

By [Kaw97, Theorem 3.1] (also [Hel97]) and our assumptions, the LC-locus LC(cD; x) (i.e., the zero-locus of the multiplier ideal $\mathcal{J}(c \cdot D)$ passing through $x$) must be a curve $C$ satisfying $1 \leq L.C \leq 2$. We can now apply Nadel's vanishing theorem to $cD$ to deduce that $H^1(X, K_X \otimes L \otimes I_C) = 0$, and so that the restriction map $H^0(X, K_X \otimes L) \to H^0(X, K_X \otimes L|_C)$ is surjective.

Consider the composition $u: L \otimes I_C \to L \otimes I_x \to O_x[2]$. Then, $u \neq 0$ if and only if $x$ is a base point of $K_X \otimes L$ which is not a base point of $K_X \otimes L|_C$. The surjectivity of the restriction map implies that $u = 0$. Hence, we get an inclusion $L \otimes I_C \hookrightarrow E$ in $\mathcal{B}$ which destabilizes $E$, if (2) is not satisfied.

\[\square\]

**REFERENCES**


