

SOME EXAMPLES OF SPACES OF STABILITY CONDITIONS ON DERIVED CATEGORIES

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1. INTRODUCTION

The notion of stability condition on a triangulated category has been introduced by Bridgeland in [8], following physical ideas of Douglas [12].

A stability condition on a triangulated category is given by abstracting the usual properties of μ -stability for sheaves on projective varieties; one introduces the notion of slope, using a group homomorphism from the Grothendieck group $K(\mathbf{T})$ of the triangulated category \mathbf{T} to \mathbb{C} and then requires that a stability condition has generalized Harder-Narasimhan filtrations and is compatible with the shift functor.

The main result of Bridgeland's paper is that these stability conditions can be described via a parameter space of stabilities. This space becomes a (possibly infinite-dimensional) manifold, called the stability manifold and denoted $\text{Stab}(\mathbf{T})$, if an extra condition (local finiteness) is assumed.

Bridgeland studied finitely dimensional slices of these spaces for the case of elliptic curves in [8] and for K3 (and abelian) surfaces in [9], but he left two cases: the curves of genus greater than one and \mathbb{P}^1 . For the curve of genus greater than one there is a simple solution, applying a technical lemma of Gorodentsev, Kuleshov and Rudakov ([14], Lemma 7.2). For completeness we include this proof in the Appendix.

For the case of \mathbb{P}^1 , the situation is slightly more involved, since in $D(\mathbb{P}^1)$ there are bounded t -structures whose heart is an abelian category of finite length (for example the one induced by the equivalence of $D(\mathbb{P}^1)$ with the derived category of representations of the Kronecker quiver [6]) and so there are degenerate stability conditions. Anyway, it is again possible to have an explicit description of $\text{Stab}(\mathbb{P}^1)$, using the classification of exceptional objects on $D(\mathbb{P}^1)$.

In this paper we generalize the study of $\text{Stab}(\mathbb{P}^1)$ to other varieties whose derived categories are generated by an exceptional sequence of sheaves, among which projective spaces and Del Pezzo surfaces. For Del Pezzo surfaces (and for some applications) see also, for example, [1], [2].

The plan of the paper is as follows. In Section 2 we give a short summary on stability conditions on triangulated categories. In Section 3 we study stability conditions associated to a complete exceptional collection on a triangulated category \mathbf{T} . After recalling some basic facts about quivers and exceptional objects, we show how to naturally associate to a complete exceptional collection with no negative homomorphisms between its objects a heart of a bounded t -structure and then a family of stability conditions having this one as heart. In this way we can define a collection of open connected subsets $\Sigma \subseteq \text{Stab}(\mathbf{T})$ of maximal dimension, parametrized by the orbits of the action of the braid group on exceptional collections (up to shifts). In Section 4 we examine some topological properties of these subsets Σ in some geometric examples. First of all, we study the derived category of the quiver P_n , with two vertices and n arrow from the first vertex to the second one (for $n = 2$ this is precisely the Kronecker quiver). In this case $\Sigma(P_n)$ is unique, simply connected and coincides with all $\text{Stab}(P_n)$. Then we apply these results to derived categories of projective spaces and Del Pezzo surfaces. In the case of \mathbb{P}^2 we also show that the open subset $\Sigma(\mathbb{P}^2)$ (which in this case is unique) is simply connected. As mentioned before, we include also an

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appendix in which stability manifolds for derived categories of smooth projective curves of positive genus are completely described.

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2. STABILITY CONDITIONS ON TRIANGULATED CATEGORIES

In this section we give a summary of Bridgeland's paper [8]. Let \mathbf{T} be a triangulated category.

Definition 2.1. A *stability condition* $\sigma = (Z, \mathcal{P})$ on \mathbf{T} consists of a group homomorphism $Z : K(\mathbf{T}) \rightarrow \mathbb{C}$, called the *central charge*, and strongly full, additive subcategories $\mathcal{P}(\phi) \subseteq \mathbf{T}$, $\phi \in \mathbb{R}$. They keep the following compatibilities: (1) for any nonzero object $E \in \mathcal{P}(\phi)$, $Z(E)/|Z(E)| = \exp(i\pi\phi)$; (2) $\forall \phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$; (3) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$, $j = 1, 2$, then $\text{Hom}_{\mathbf{T}}(A_1, A_2) = 0$; (4) for any nonzero object $E \in \mathbf{T}$ there is a finite sequence of real numbers $\phi_1 > \dots > \phi_n$ and a collection of triangles $E_{j-1} \rightarrow E_j \rightarrow A_j$ with $A_j \in \mathcal{P}(\phi_j)$, for all j , $E_0 = 0$ and $E_n = E$.

The collection of exact triangles in Definition 2.1 (4) is called the *Harder-Narasimhan filtration* of E (*HN filtration* for short). Note that the HN filtration is unique up to isomorphisms. We write $\phi_{\sigma}^{+}(E) := \phi_1$, $\phi_{\sigma}^{-}(E) := \phi_n$, and $m_{\sigma}(E) := \sum_j |Z(A_j)|$. From the definition, each subcategory $\mathcal{P}(\phi)$ is extension-closed and abelian. Its nonzero objects are said to be *semistable* of phase ϕ in σ , and the minimal objects (classically called simple objects) are said to be *stable*.

For any interval $I \subseteq \mathbb{R}$, $\mathcal{P}(I)$ is defined to be the extension-closed subcategory of \mathbf{T} generated by the subcategories $\mathcal{P}(\phi)$, for $\phi \in I$. Bridgeland proved that, for all $\phi \in \mathbb{R}$, $\mathcal{P}((\phi, \phi + 1])$ is the heart of a bounded t -structure on \mathbf{T} . The category $\mathcal{P}((0, 1])$ is called the *heart* of σ .

Remark 2.2. Let $H := \{z \in \mathbb{C} : z = |z| \exp(i\pi\phi), 0 < \phi \leq 1\}$. If $\mathbf{A} \subseteq \mathbf{T}$ is the heart of a bounded t -structure and moreover it is an abelian category of finite length (i.e. artinian and noetherian) with a finite number of minimal objects, then by [8, Proposition 5.3], a group homomorphism $Z : K(\mathbf{A}) \rightarrow \mathbb{C}$ such that $Z(E) \in H$ for all $E \in \mathbf{A}$ (such a group homomorphism is called a *stability function*), extends to a unique stability condition on \mathbf{T} .

Lemma 2.3. *Let (Z, \mathcal{P}) be a stability condition on \mathbf{T} . Assume that \mathbf{A} is a full abelian subcategory of $\mathcal{P}((0, 1])$ and the heart of a bounded t -structure on \mathbf{T} . Then $\mathbf{A} = \mathcal{P}((0, 1])$.*

Proof. By the definition of bounded t -structure (cfr., for example, [8, Lemma 3.2]), if an object E is in $\mathcal{P}((0, 1])$ and not in \mathbf{A} , then there is a nonzero morphism either $A[k] \rightarrow E$ or $E \rightarrow A[-k]$, with $k > 0$, $A \in \mathbf{A}$. But for all $j \in \mathbb{Z}$, $\mathbf{A}[j]$ is an abelian subcategory of $\mathcal{P}((0, 1])[j]$, which leads to a contradiction. \blacksquare

A stability condition is called *locally-finite* if there exists some $\varepsilon > 0$ such that each quasi-abelian subcategory $\mathcal{P}((\phi - \varepsilon, \phi + \varepsilon))$ is of finite length. In this way $\mathcal{P}(\phi)$ has finite length so that every object in $\mathcal{P}(\phi)$ has a finite Jordan-Hölder filtration into stable factors of the same phase. The set of stability conditions which are locally-finite will be denoted by $\text{Stab}(\mathbf{T})$.

By [8, Proposition 8.1] there is a natural topology on $\text{Stab}(\mathbf{T})$ defined by the generalized metric (i.e. it may be infinite)

$$d(\sigma_1, \sigma_2) := \sup_{0 \neq E \in \mathbf{T}} \left\{ |\phi_{\sigma_2}^{+}(E) - \phi_{\sigma_1}^{+}(E)|, |\phi_{\sigma_2}^{-}(E) - \phi_{\sigma_1}^{-}(E)|, \left| \log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \right| \right\} \in [0, \infty]. \quad (1)$$

We call $\text{Stab}(\mathbf{T})$ the *stability manifold* associated to \mathbf{T} .

Theorem 2.4. [8, Theorem 1.2] *For each connected component $\Sigma \subseteq \text{Stab}(\mathbf{T})$ there is a linear subspace $V(\Sigma) \subseteq (K(\mathbf{T}) \otimes \mathbb{C})^{\vee}$ with a well-defined linear topology such that the natural map $\mathcal{Z} : \Sigma \rightarrow V(\Sigma)$, which maps a stability condition (Z, \mathcal{P}) to its central charge Z , is a local homeomorphism. In particular, if $K(\mathbf{T}) \otimes \mathbb{C}$ is finite dimensional, then Σ is a finite dimensional complex manifold.*

For later use, we remind that $V(\Sigma)$ is defined as the set of $W \in (K(\mathbf{T}) \otimes \mathbb{C})^\vee$ such that

$$\|W\|_\sigma := \sup \left\{ \frac{|W(E)|}{|Z(E)|} : E \text{ is } \sigma\text{-semistable} \right\} < \infty,$$

where $\sigma = (Z, \mathcal{P})$ is any stability condition in Σ .

Remark 2.5. [8, Lemma 8.2] The generalized metric space $\text{Stab}(\mathbf{T})$ carries a right action of the group $\widetilde{GL^+}(2, \mathbb{R})$, the universal cover of $GL^+(2, \mathbb{R})$, and a left action by isometries of the group $\text{Auteq}(\mathbf{T})$ of exact autoequivalences of \mathbf{T} . The second action is defined in the natural way. For the first action, let $(G, f) \in \widetilde{GL^+}(2, \mathbb{R})$, with $G \in GL^+(2, \mathbb{R})$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing map, periodic of period 1 such that $Ge^{2i\pi\phi}/|Ge^{2i\pi\phi}| = e^{2i\pi f(\phi)}$, for all $\phi \in \mathbb{R}$. Then (G, f) maps $(Z, \mathcal{P}) \in \text{Stab}(\mathbf{T})$ to $(G^{-1} \circ Z, \mathcal{P} \circ f)$.

3. QUIVERS AND EXCEPTIONAL OBJECTS

3.1. Quivers and algebras. In this subsection we give a quick review of some basic facts about finite dimensional algebras over \mathbb{C} and we start studying stability conditions on their derived categories. For more details see [5].

A quiver is a directed graph, possibly with multiple arrows and loops. In this paper we deal only with finite quivers, that is those which have a finite number of vertices and arrows. If Q is a quiver, we define its path-algebra $\mathbb{C}Q$ as follows. It is an algebra over \mathbb{C} , which as a vector space has a basis consisting of the paths

$$\bullet \rightarrow \dots \rightarrow \bullet$$

in Q . Multiplication is given by composition of paths if the paths are composable in this way, and zero otherwise. Corresponding to each vertex x there is a path of length zero giving rise to idempotent basis elements e_x . Clearly $\mathbb{C}Q$ is finitely generated as an algebra over \mathbb{C} if and only if Q has only finitely many vertices and arrows, and finite dimensional as a vector space if and only if in addition it has no loops.

A representation of a quiver Q associates to each vertex x of Q a vector space V_x , and to each arrow $x \rightarrow y$ a linear transformation $V_x \rightarrow V_y$ between the corresponding vector spaces. The dimension vector α of such a representation is a vector of integers having length equal to the number of vertices of the quiver given by $\alpha_x = \dim V_x$. There is a natural one-to-one correspondence between representations of Q and $\mathbb{C}Q$ -modules. If Q is finite without loops, then the simple modules correspond to vertices of the quivers (that is to representations which consist of a one dimensional vector space at a vertex and a zero dimensional vector space at any other vertex) and the indecomposable projective modules are of the form $P_x = \mathbb{C}Q \cdot e_x$.

The importance of quivers in the theory of representations of finite dimensional algebras is illustrated by the following theorem (cfr. [5, Proposition 4.1.7]) due to Gabriel.

Theorem 3.1. *Every finite dimensional basic algebra over \mathbb{C} (i.e. every simple module is one dimensional over \mathbb{C}) is the quotient of a path-algebra $\mathbb{C}Q$ of a quiver Q modulo an ideal contained in the ideal of paths of length at least two. In particular, if $\mathbb{C}Q$ is finite dimensional, there is a bijection between the simple modules for the algebra and the simple $\mathbb{C}Q$ -modules.*

Definition 3.2. Let I be a two-sided ideal contained in the ideal of paths of length at least two of the path-algebra of a quiver Q . We call the pair (Q, I) a quiver with relations (sometimes, with abuse of notation, we will forget the ideal I). The path-algebra of a quiver with relations is the algebra $\mathbb{C}Q/I$.

As for quivers, we can define representations of a quiver with relations Q . Again there is a natural one-to-one correspondence between representations of Q and modules over its path-algebra.

Example 3.3. The quiver P_n ($n \geq 1$) contains two vertices and n arrows from the first to the second vertex. For example,

$$P_2 : \bullet \rightrightarrows \bullet.$$

Example 3.4. The quiver with relations T_N ($N \geq 1$) contains $N + 1$ vertices X_0, \dots, X_N and $N(N + 1)$ arrows $\phi_i^j : X_i \rightarrow X_{i+1}$ ($i = 0, \dots, N - 1, j = 0, \dots, N$). The relations are $\phi_{i+1}^j \phi_i^k = \phi_{i+1}^k \phi_i^j$. Note that T_1 coincides with P_2 .

In the next subsection we will see the connections between exceptional objects in derived categories and quivers with relations. We conclude now by examining stability conditions (Z, \mathcal{P}) on the derived category of a finite dimensional algebra A for which $\mathcal{P}((0, 1]) = \mathbf{mod}\text{-}A$, where $\mathbf{mod}\text{-}A$ denotes the category of finitely generated (right) A -modules.

Lemma 3.5. *Let A be a finite-dimensional algebra over \mathbb{C} with simple modules $\{L_0, \dots, L_n\}$ and let (Z, \mathcal{P}) be a stability condition on $D(A) := D^b(\mathbf{mod}\text{-}A)$. Assume that $L_0, \dots, L_n \in \mathcal{P}((0, 1])$. Then $\mathbf{mod}\text{-}A = \mathcal{P}((0, 1])$ and L_j is stable, for all $j = 0, \dots, n$.*

Proof. Since $\mathbf{mod}\text{-}A$ is the extension closed subcategory of $D(A)$ generated by L_0, \dots, L_n , then $\mathbf{mod}\text{-}A$ is an abelian subcategory of $\mathcal{P}((0, 1])$; so, the first part follows immediately from Lemma 2.3. Now the second statement is clear, since L_j is a minimal object of $\mathbf{mod}\text{-}A$. \blacksquare

Remark 3.6. Note that, as observed in [2, Theorem 3], in the situation of the previous lemma, an object of $D(A)$ is (semi)stable if and only if it is a shift of a θ -(semi)stable A -module in the sense of King [16]. In particular one can construct moduli spaces of semistable objects having fixed dimension vector.

3.2. Exceptional objects. Some references for this subsection are [6], [15], [17]. Let \mathbf{T} be a triangulated category linear over \mathbb{C} and of finite type, i.e. for any two objects $A, B \in \mathbf{T}$ the \mathbb{C} -vector space $\bigoplus_{k \in \mathbb{Z}} \text{Hom}^k(A, B)$ is finite-dimensional. Following [6] we introduce the following notation for the graded complex of \mathbb{C} vector spaces with trivial differential

$$\text{Hom}^\bullet(A, B) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}^k(A, B)[-k],$$

where $A, B \in \mathbf{T}$, $\text{Hom}^k(A, B) = \text{Hom}(A, B[k])$. When \mathbf{T} is the derived category of an abelian category, $\text{Hom}^\bullet(A, B)$ is quasi-isomorphic to $\mathbf{R}\text{Hom}(A, B)$.

Definition 3.7. (i) An object $E \in \mathbf{T}$ is called *exceptional* if it satisfies

$$\begin{aligned} \text{Hom}^i(E, E) &= 0, & \text{for } i \neq 0, \\ \text{Hom}^0(E, E) &= \mathbb{C}. \end{aligned}$$

(ii) An ordered collection of exceptional objects $\{E_0, \dots, E_n\}$ is called *exceptional* in \mathbf{T} if it satisfies

$$\text{Hom}^\bullet(E_i, E_j) = 0, \quad \text{for } i > j.$$

We call an exceptional collection of two objects an *exceptional pair*.

Definition 3.8. Let (E, F) an exceptional pair. We define objects $\mathcal{L}_E F$ and $\mathcal{R}_F E$ (which we call *left mutation* and *right mutation* respectively) with the aid of distinguished triangles

$$\begin{aligned} \mathcal{L}_E F &\rightarrow \text{Hom}^\bullet(E, F) \otimes E \rightarrow F, \\ E &\rightarrow \text{Hom}^\bullet(E, F)^* \otimes F \rightarrow \mathcal{R}_F E, \end{aligned}$$

where $V[k] \otimes E$ (with V vector space) denotes an object isomorphic to the direct sum of $\dim V$ copies of the object $E[k]$. Note that under duality of vector spaces the grading changes sign.

A *mutation* of an exceptional collection $\mathcal{E} = \{E_0, \dots, E_n\}$ is defined as a mutation of a pair of adjacent objects in this collection:

$$\begin{aligned} \mathcal{R}_i \mathcal{E} &= \{E_0, \dots, E_{i-1}, E_{i+1}, \mathcal{R}_{E_{i+1}} E_i, E_{i+2}, \dots, E_n\}, \\ \mathcal{L}_i \mathcal{E} &= \{E_0, \dots, E_{i-1}, \mathcal{L}_{E_i} E_{i+1}, E_i, E_{i+2}, \dots, E_n\}, \end{aligned}$$

$i = 0, \dots, n - 1$. We can do mutations again in the mutated collection. We call any composition of mutations an *iterated mutation*.

Proposition 3.9. [6] (i) A mutation of an exceptional collection is an exceptional collection.
 (ii) If an exceptional collection generates \mathbf{T} then the mutated collection also generates \mathbf{T} .
 (iii) The following relations hold:

$$\mathcal{R}_i \mathcal{L}_i = \mathcal{L}_i \mathcal{R}_i = 1 \quad \mathcal{R}_i \mathcal{R}_{i+1} \mathcal{R}_i = \mathcal{R}_{i+1} \mathcal{R}_i \mathcal{R}_{i+1} \quad \mathcal{L}_i \mathcal{L}_{i+1} \mathcal{L}_i = \mathcal{L}_{i+1} \mathcal{L}_i \mathcal{L}_{i+1}.$$

The last relations, together with the obvious commutativity $\mathcal{R}_i \mathcal{R}_j = \mathcal{R}_j \mathcal{R}_i$ for $j - i \neq \pm 1$, could be rephrased by saying that there is an action of the braid group A_{n+1} of $n + 1$ strings on the set of exceptional collections.

Definition 3.10. Let $\mathcal{E} = \{E_0, \dots, E_n\}$ be an exceptional collection. We call \mathcal{E}

- strong, if $\text{Hom}^k(E_i, E_j) = 0$ for all i and j , with $k \neq 0$;
- Ext, if $\text{Hom}^{\leq 0}(E_i, E_j) = 0$ for all $i \neq j$;
- regular, if $\text{Hom}^k(E_i, E_j) \neq 0$ for at most one $k \geq 0$, for all i and j ;
- orthogonal, if $\text{Hom}^k(E_i, E_j) = 0$ for all $i \neq j$ and k ;
- complete, if \mathcal{E} generates \mathbf{T} by shifts and extensions.

The relation between strong exceptional collections and finite dimensional algebras is contained in the following result due to Bondal.

Theorem 3.11. Let \mathbf{T} be the bounded derived category of an abelian category. Assume that \mathbf{T} is generated by a strong exceptional collection $\{E_0, \dots, E_n\}$. Then, if we set $E = \bigoplus E_i$ and $A = \text{Hom}(E, E)$, \mathbf{T} is equivalent to the bounded derived category of finite dimensional modules over the algebra A .

Proof. Define the functor $\Phi : \mathbf{T} \rightarrow D(A)$ as the derived functor $\Phi(Y) = \mathbf{R}\text{Hom}(E, Y)$, where $Y \in \mathbf{T}$, with the natural action of A on that complex. For the proof that Φ is actually an equivalence see [6, Theorem 6.2]. \blacksquare

Example 3.12. [3] [15] Let $\mathbf{T} = D(\mathbb{P}^N) := D^b(\mathbf{Coh}(\mathbb{P}^N))$. Then a complete strong exceptional collection is given by $\{\mathcal{O}, \dots, \mathcal{O}(N)\}$. The corresponding algebra is given by the quiver with relations T_N of Example 3.4.

Example 3.13. [17] Let $\mathbf{T} = D(S) := D^b(\mathbf{Coh}(S))$, where S is a Del Pezzo surface, i.e. a smooth projective surface whose anticanonical class is ample. Then there exists a complete strong exceptional collection and all exceptional collections are obtained from this collection by iterated mutations.

3.3. Stability conditions on triangulated categories generated by an exceptional collection. In this subsection we study stability conditions on triangulated categories generated by Ext-exceptional collections. Given a subcategory S of \mathbf{T} , we denote by $\langle S \rangle$ the extension-closed subcategory of \mathbf{T} generated by S , and by $\text{Tr}(S)$ the minimal full triangulated subcategory containing S and closed by isomorphisms.

Lemma 3.14. Let $\{E_0, \dots, E_n\}$ be a complete Ext-exceptional collection on \mathbf{T} . Then $\langle E_0, \dots, E_n \rangle$ is the heart of a bounded t -structure on \mathbf{T} .

Proof. We proceed by induction on n . If $n = 0$ there is nothing to prove. Indeed $\mathbf{T} \cong D^b(\mathbf{C}\text{-}\mathbf{vect})$, Assume $n > 0$. Then consider the full triangulated subcategory $\text{Tr}(E_1, \dots, E_n)$ of \mathbf{T} . This is an admissible subcategory [6, Theorem 3.2] and its right orthogonal is $\text{Tr}(E_0)$. Moreover

$$\text{Tr}(E_1, \dots, E_n) \xrightarrow{i^*} \mathbf{T} \xrightarrow{j^*} \text{Tr}(E_0)$$

is an exact triple of triangulated categories [7, Proposition 1.6].

By [4, §1.4] any pair of t -structures on $\text{Tr}(E_1, \dots, E_n)$ and $\text{Tr}(E_0)$ determines a unique compatible t -structure on \mathbf{T} given by

$$\begin{aligned} \mathbf{T}^{\leq 0} &= \{F \in \mathbf{T} : j^*F \in \text{Tr}(E_0)^{\leq 0}, i^*F \in \text{Tr}(E_1, \dots, E_n)^{\leq 0}\} \\ \mathbf{T}^{\geq 0} &= \{F \in \mathbf{T} : j^*F \in \text{Tr}(E_0)^{\geq 0}, i^!F \in \text{Tr}(E_1, \dots, E_n)^{\geq 0}\}, \end{aligned}$$

where i^* and $i^!$ are respectively the left and right adjoint to i_* . More explicitly, if F decomposes as

$$A \rightarrow F \rightarrow B$$

with $A \in \mathrm{Tr}(E_1, \dots, E_n)$, $B \in \mathrm{Tr}(E_0)$, then $i^!F = A$; if F decompose as

$$B' \rightarrow F \rightarrow A'$$

with $A' \in \mathrm{Tr}(E_1, \dots, E_n)$, $B' \in \mathrm{Tr}(\mathcal{R}_{n-1} \dots \mathcal{R}_0 E_0)$, then $i^*F = A'$.

By induction we can choose t -structures on $\mathrm{Tr}(E_0)$ and $\mathrm{Tr}(E_1, \dots, E_n)$ having hearts $\langle E_0 \rangle$ and $\langle E_1, \dots, E_n \rangle$ respectively.

We want to prove that $\langle E_0, \dots, E_n \rangle = \mathbf{T}^{\leq 0} \cap \mathbf{T}^{\geq 0} =: \mathbf{A}$. Clearly E_1, \dots, E_n belong to \mathbf{A} . Moreover, by mutating E_0 , since the exceptional collection is Ext, we have $i^*E_0 \in \mathrm{Tr}(E_1, \dots, E_n)^{\leq 0}$. Hence $\langle E_0, \dots, E_n \rangle$ is a full subcategory of \mathbf{A} . If $F \in \mathbf{A}$, $F \notin \langle E_0, \dots, E_n \rangle$, we can filter F as

$$A \rightarrow F \rightarrow B \tag{2}$$

as before. By construction $B \in \langle E_0 \rangle$. We want to prove that $A \in \langle E_1, \dots, E_n \rangle$. Assume the contrary. From the triangle (2) we get

$$0 \rightarrow H^0(A) \rightarrow H^0(F) \rightarrow H^0(B) \rightarrow H^1(A) \rightarrow 0.$$

Since $A \in \mathrm{Tr}(E_1, \dots, E_n)$, by induction we can filter it as

$$C \rightarrow A \rightarrow D,$$

with $C \in \langle E_1, \dots, E_n \rangle$ and $D \in \langle E_1, \dots, E_n \rangle[-1]$. But then $H^0(A) \cong C$ and $H^1(A) \cong D[1]$. This means that we have a map $B \rightarrow D[1]$, a contradiction.

The fact that the glued t -structure is bounded is now clear. \blacksquare

Corollary 3.15. *Let $\mathcal{E} = \{E_0, \dots, E_n\}$ be a complete exceptional collection on \mathbf{T} such that, for some $i < j$, $\mathrm{Hom}^{\leq 0}(E_i, E_j) = 0$. Then $\langle E_i, E_j \rangle$ is a full abelian subcategory of \mathbf{T} .*

Lemma 3.16. *Let $\{E_0, \dots, E_n\}$ be a complete Ext-exceptional collection on \mathbf{T} and let (Z, \mathcal{P}) be a stability condition on \mathbf{T} . Assume $E_0, \dots, E_n \in \mathcal{P}((0, 1])$. Then $\mathbf{Q} := \langle E_0, \dots, E_n \rangle = \mathcal{P}((0, 1])$ and E_j is stable, for all $j = 0, \dots, n$.*

Proof. First of all notice that E_0, \dots, E_n are the minimal objects of \mathbf{Q} . Indeed, let $0 \neq A \hookrightarrow E_i$ be a subobject in \mathbf{Q} . Then by definition there exists an inclusion $E_j \hookrightarrow A$, for some j . But, since $\{E_0, \dots, E_n\}$ is an Ext-exceptional collection, then j should be equal to i . Hence the composite map $E_j \hookrightarrow A \hookrightarrow E_i$ is an isomorphism and $A \cong E_i$. The proof is then the same as Lemma 3.5. \blacksquare

If two exceptional objects have at most one nontrivial Hom^k , one can say more.

Proposition 3.17. *Let $\{E_0, \dots, E_n\}$ be a complete Ext-exceptional collection on \mathbf{T} and let $\sigma = (Z, \mathcal{P})$ be a stability condition on \mathbf{T} such that $E_0, \dots, E_n \in \mathcal{P}((0, 1])$. Fix $i < j$. Then σ induces a stability condition σ_{ij} on $\mathrm{Tr}(E_i, E_j)$ in such a way that every (semi)stable object in σ_{ij} with phase ϕ corresponds to a (semi)stable object in σ with the same phase. Moreover, assume that $\mathrm{Hom}^1(E_i, E_j) \neq 0$ and $\mathrm{Hom}^k(E_i, E_j) = 0$, for all $k \neq 1$. Then, if*

$$\phi(E_j) \leq \phi(E_i),$$

$\mathcal{R}_{E_j}E_i$ and $\mathcal{L}_{E_i}E_j$ are semistable and, if $\phi(E_j) < \phi(E_i)$, they are stable.

Proof. Consider the triangulated category $\mathrm{Tr}(E_i, E_j)$ generated by E_i and E_j . Then Z defines a stability function on the abelian category $\langle E_i, E_j \rangle$, and so a stability condition on $\mathrm{Tr}(E_i, E_j)$, by Remark 2.2.

Let S be a semistable object in $\mathrm{Tr}(E_i, E_j)$. We can assume $S \in \langle E_i, E_j \rangle$ and $\phi(E_j) \leq \phi(E_i)$. Let

$$0 \rightarrow A \rightarrow S \rightarrow B \rightarrow 0$$

objects (up to shifts). But notice that in general $\Theta_{\mathcal{E}}$ is not the subset of $\text{Stab}(\mathbf{T})$ consisting of stability conditions in which E_0, \dots, E_n are stable (see Remark 4.8).

Lemma 3.18. $\Theta_{\mathcal{E}} \subseteq \text{Stab}(\mathbf{T})$ is an open, connected and simply connected $(n+1)$ -dimensional submanifold.

Proof. Before proceeding with the proof we need some notations. Let $\mathcal{F}_s = \{F_0, \dots, F_s\}$, $s > 0$, be an exceptional collection. First of all, define, for $i < j$,

$$k_{i,j}^{\mathcal{F}_s} := \begin{cases} +\infty, & \text{if } \text{Hom}^k(F_i, F_j) = 0, \text{ for all } k; \\ \min \left\{ k : \text{Hom}^k(F_i, F_j) \neq 0 \right\}, & \text{otherwise.} \end{cases}$$

Then define inductively $\alpha_i^{\mathcal{F}_s} \in \mathbb{Z} \cup \{+\infty\}$ in this way: set $\alpha_s^{\mathcal{F}_s} = 0$ and, for $i < s$,

$$\alpha_i^{\mathcal{F}_s} := \min_{j>i} \left\{ k_{i,j}^{\mathcal{F}_s} + \alpha_j^{\mathcal{F}_s} \right\} - (s - i - 1),$$

where the minimum is taken over $\mathbb{Z} \cup \{+\infty\}$.

Consider \mathbb{R}^{n+1} with coordinates ϕ_0, \dots, ϕ_n . Let $\mathcal{F}_s := \{E_{l_0}, \dots, E_{l_s}\} \subseteq \{E_0, \dots, E_n\}$, $s > 0$. Define $R^{\mathcal{F}_s}$ as the relation $\phi_{l_0} < \phi_{l_s} + \alpha_0^{\mathcal{F}_s}$. Finally define

$$\mathcal{C}_{\mathcal{E}} := \left\{ (m_0, \dots, m_n, \phi_0, \dots, \phi_n) \in \mathbb{R}^{2(n+1)} : \begin{array}{l} \bullet m_i > 0, \text{ for all } i \\ \bullet R^{\mathcal{F}_s}, \text{ for all } \mathcal{F}_s \subseteq \mathcal{E}, s > 0 \end{array} \right\}.$$

Clearly $\mathcal{C}_{\mathcal{E}}$ is connected and simply connected.

Define a map $\rho : \Theta_{\mathcal{E}} \rightarrow \mathcal{C}_{\mathcal{E}}$ by $m_i(\rho(\sigma)) := Z(E_i)/|Z(E_i)|$, $\phi_i(\rho(\sigma)) := \phi_{\sigma}(E_i)$, for $i = 0, \dots, n$, where $\sigma = (Z, \mathcal{P})$. We want to prove that this map is an homeomorphism. By definition of $\Theta_{\mathcal{E}}$, ρ is injective. Moreover it is straightforward to check that it is also surjective.

Consider the abelian category \mathbf{Q}_p , for p_0, \dots, p_n integral numbers such that the complete exceptional collection $\{E_0[p_0], \dots, E_n[p_n]\}$ is Ext. Let $\sigma_p \in \Theta_{\mathcal{E}}$ be the stability condition defined by setting $Z_p(E_j[p_j]) = i$, for all j and let $\Gamma_p \subseteq \text{Stab}(\mathbf{T})$ be the connected component containing σ_p . First of all notice that the linear subspace $V(\Gamma_p)$ of Theorem 2.4 is all $(K(\mathbf{T}) \otimes \mathbb{C})^{\vee}$. Indeed, for all $W \in (K(\mathbf{T}) \otimes \mathbb{C})^{\vee}$,

$$\begin{aligned} \|W\|_{\sigma_p} &= \sup \left\{ \frac{|W(E)|}{|Z_p(E)|} : E \text{ is } \sigma_p\text{-semistable} \right\} \\ &= \sup \left\{ \frac{|W(E)|}{|Z_p(E)|} : E \in \mathbf{Q}_p \right\} \\ &= \sup \left\{ \frac{|a_0 W(E_0[p_0]) + \dots + a_n W(E_n[p_n])|}{a_0 + \dots + a_n} : a_0, \dots, a_n \geq 0 \right\} \\ &\leq \sup \left\{ \frac{a_0 |W(E_0[p_0])| + \dots + a_n |W(E_n[p_n])|}{a_0 + \dots + a_n} : a_0, \dots, a_n \geq 0 \right\} \\ &< \infty. \end{aligned}$$

Hence Γ_p is a manifold of dimension $(n+1)$. Since the map Z of Theorem 2.4 is a local homeomorphism, to prove the lemma is sufficient to show that $\Theta_{\mathcal{E}}$ is contained in Γ_p . But, by definition of the generalized metric (1), all stability conditions with heart \mathbf{Q}_p are in Γ_p and so is in Γ_p the open subset U_p consisting of stability conditions which have, up to the action of $\widetilde{GL^+(2, \mathbb{R})}$, \mathbf{Q}_p as heart.

Now, let l be an index such that $\text{Hom}^0(E_s[p_j], E_s[p_l+1]) = 0$, for all $s < l$ (for example $l = 0$ always works). Then if we set $p'_l := p_l + 1$ and $p'_j := p_j$, for $j \neq l$, then the exceptional collection $\{E_0[p'_0], \dots, E_n[p'_n]\}$ is still Ext and $U'_p \cap U_p \neq \emptyset$. Indeed, the stability condition with heart \mathbf{Q}_p defined by $Z(E_l[p_l]) = i$, $Z(E_j[p_j]) = -1$, for $j \neq l$ is in the $\widetilde{GL^+(2, \mathbb{R})}$ -orbit of the stability condition with heart $\mathbf{Q}_{p'}$, defined by $Z(E_l[p'_l]) = -1$, $Z(E_j[p'_j]) = i$, for $j \neq l$.

Finally, given two collection of integers p and q such that \mathcal{E}_p and \mathcal{E}_q are Ext, it is always possible to find a third collection r such that r is obtained from p and q by successively adding 1 to some integer corresponding to an index l as before. Hence $\Theta_{\mathcal{E}} \subseteq \Gamma_p$, for some (all) p , as wanted. \blacksquare

Assume that \mathcal{E} is regular and that all its iterated mutations are again regular. Define $\Sigma_{\mathcal{E}}$ as the union of the open subsets $\Theta_{\mathcal{F}}$ over all iterated mutations \mathcal{F} of \mathcal{E} . When the triangulated

category \mathbf{T} is *constructible*, i.e. all complete exceptional collections can be obtained, up to shifts, by iterated mutations of a single complete exceptional collection, we simply denote the previous open subset by $\Sigma(\mathbf{T})$.

Corollary 3.19. $\Sigma_{\mathcal{E}} \subseteq \text{Stab}(\mathbf{T})$ is an open and connected $(n+1)$ -dimensional submanifold.

Proof. It sufficient to show that, for a single mutation \mathcal{F} of \mathcal{E} , $\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{E}}$ is nonempty. We can restrict to consider $\mathcal{F} = \mathcal{R}_j \mathcal{E}$, $j \in \{0, \dots, n-1\}$, with (E_j, E_{j+1}) not an orthogonal pair. Fix integers p_0, \dots, p_n such that $\text{Hom}^k(E_l[p_l], E_t[p_t]) = 0$, for all $k \leq 1$ and all pair (l, t) , $l \neq t$ besides $(j, j+1)$, where $\text{Hom}^1(E_j[p_j], E_{j+1}[p_{j+1}]) \neq 0$. In particular $\{E_0[p_0], \dots, E_n[p_n]\}$ is Ext. Fix $z_0, \dots, z_n \in H$ such that $z_l = i$ for $l \neq j, j+1$, $z_j = -1$ and $z_{j+1} = 1+i$ and consider the abelian category \mathbf{Q}_p . Let σ be the stability condition constructed by these data. Then $\sigma \in \Theta_{\mathcal{F}} \cap \Theta_{\mathcal{E}}$. Indeed, the exceptional collection

$$\{E_0[p_0], \dots, E_{j+1}[p_{j+1} + 1], \mathcal{R}_{E_{j+1}} E_j[p_j - 1], \dots, E_n[p_n]\}$$

is still Ext and consists of σ -stable objects with phases in the interval $[-1/2, 1/2)$, by Proposition 3.17. But then, using the $\widehat{GL}^+(2, \mathbb{R})$ -action, Lemma 3.16 implies that σ is in $\Theta_{\mathcal{F}}$. \blacksquare

4. EXAMPLES

In this section we examine some spaces of stability conditions. First of all, using the description of exceptional objects on the category of representations of quivers without loops given in [11, 19], we study the space $\text{Stab}(P_n)$, which in particular for $n=2$ is the space of stability conditions on the derived category of \mathbb{P}^1 . Then, using this, we describe some topological properties of the open subsets Σ , defined in the previous section, for projective spaces and on Del Pezzo surfaces (for Del Pezzo surfaces see also [2]).

4.1. $\text{Stab}(P_n)$. Let P_n be the quiver defined in the Example 3.3 and \mathcal{Q}_n the abelian category of its finite dimensional representations. Since the case $n=0$ is trivial, assume $n > 0$. Set $\{S_i\}_{i \in \mathbb{Z}}$ the family of exceptional objects on $D(P_n)$, where $S_0[1]$ and S_1 are the minimal objects in \mathcal{Q}_n and the others exceptional objects are defined by

$$\begin{aligned} S_i &:= \mathcal{L}_{S_{i+1}} S_{i+2}, & i < 0, \\ S_i &:= \mathcal{R}_{S_{i-1}} S_{i-2}, & i \geq 2. \end{aligned}$$

According to [11, 19] these are (up to shifts) the only exceptional objects in $D(P_n)$. Note that, since \mathcal{Q}_n is an abelian category of dimension 1, $S_{\leq 0}[1], S_{> 0} \in \mathcal{Q}_n$ unless $n=1$. In fact the case $n=1$ is also somehow degenerate: indeed in that case three mutations are equal to a shift and so there are effectively only three exceptional objects up to shifts. The main results of this subsection (Lemma 4.2 and Theorem 4.5) still hold true for $n=1$; the proofs are a little bit different but easier. Hence we leave them to the reader and in the following we assume $n \geq 2$.

Lemma 4.1. *If $i < j$, then*

- $\text{Hom}^k(S_i, S_j) \neq 0$ only if $k=0$;
- $\text{Hom}^k(S_j, S_i) \neq 0$ only if $k=1$.

In particular the pair (S_k, S_{k+1}) is a complete strong exceptional collection.

Lemma 4.2. *In every stability condition on $D(P_n)$ there exists a stable exceptional pair (E, F) .*

Proof. First of all note that, since $\dim \mathcal{Q}_n = 1$, each object of $D(P_n)$ is isomorphic to a finite direct sum of shifts of objects of \mathcal{Q}_n . So, if an object is stable, some shift of it must belong to \mathcal{Q}_n .

Let L be an exceptional object of $D(P_n)$. We can assume L to be $S_0[1]$ in \mathcal{Q}_n . Suppose that L is not semistable. Then there exists a destabilizing triangle (the last triangle of the HN filtration)

$$X \rightarrow L \xrightarrow{f} A \rightarrow X[1],$$

with A semistable and

$$\text{Hom}^{\leq 0}(X, A) = 0. \tag{4}$$

From the previous remark, we can assume

$$\begin{aligned} A &= B_0[0] \oplus B_1[1], & B_i &\in \mathcal{Q}_n \\ f &= f_0 + f_1, & f_0 &\in \text{Hom}(L, B_0), \quad f_1 \in \text{Ext}_{\mathcal{Q}_n}^1(L, B_1). \end{aligned}$$

Moreover, $f_i = 0$ if and only if $B_i = 0$, for $i = 0, 1$.

But, if $f_0 \neq 0$, then B_0 has a direct factor isomorphic to L , which is of course not possible. So, $B_0 = 0$ and the destabilizing triangle is obtained by the extension in \mathcal{Q}_n

$$0 \rightarrow B_1 \rightarrow X \rightarrow L \rightarrow 0 \quad (5)$$

corresponding to f_1 , with the conditions $\text{Hom}(X, B_1) = \text{Ext}^1(X, B_1) = 0$. Applying to (5) the functor $\text{Hom}(\bullet, B_1)$ we get $\text{Ext}^1(B_1, B_1) = \text{Hom}(L, B_1) = 0$. If B_1 is indecomposable, then by [11, 19] B_1 must be an exceptional object (i.e. $\text{Hom}(B_1, B_1) = \mathbb{C}$). Otherwise, if B_1 is not indecomposable, then again by [11, 19], since there are no orthogonal exceptional pairs¹, B_1 must be of the form $E^{\oplus i}$ for some exceptional object $E \in \mathcal{Q}_n$. But then, by Lemma 4.1, $\text{Ext}^1(B_1, L) = 0$. Applying to (5) the functor $\text{Hom}(\bullet, L)$ we get $\text{Ext}^1(X, L) = 0$. Applying the functor $\text{Hom}(X, \bullet)$ we get $\text{Ext}^1(X, X) = 0$. Again $X \cong F^{\oplus j}$, for some exceptional object F , and so (E, F) is an exceptional pair. But also F is semistable. Indeed, suppose it is not; then the HN filtration for L continues

$$R' \rightarrow X \rightarrow A' \rightarrow R'[1],$$

with

$$\begin{aligned} \text{Hom}^{\leq 0}(R', A') &= 0, \\ \text{Hom}^{\leq 0}(A', A) &= 0. \end{aligned} \quad (6)$$

Now, proceeding as before, A' and R' are direct sums of exceptional objects. But condition (6) implies that A' must be X , by Lemma 4.1. Hence F is semistable.

To conclude we only have to prove that E and F are actually stable. But, by the first part of the proof, if E is not stable (the proof for F is the same), then all its stable factors must be isomorphic to a single object K , which of course implies $K \cong E$. \blacksquare

Corollary 4.3. *In every stability condition on $D(\mathbb{P}^1)$ there exists an integer k such that the line bundles $\mathcal{O}(k)$ and $\mathcal{O}(k+1)$ are stable.*

Let Θ_k , $k \in \mathbb{Z}$, be the open connected and simply connected subset of $\text{Stab}(P_n) := \text{Stab}(D(P_n))$ defined in the previous section, consisting of stability conditions which are, up to the action of $\widetilde{GL^+(2, \mathbb{R})}$, degenerate or non-degenerate for the exceptional pair (S_k, S_{k+1}) . By Lemma 3.16, Θ_k coincides with the subset of $\text{Stab}(P_n)$ consisting of stability conditions in which S_k and S_{k+1} are stable. Moreover, by Lemma 4.2, $\text{Stab}(P_n)$ is the union over \mathbb{Z} of its subsets Θ_k , i.e. it coincides with its open subset $\Sigma(P_n)$ defined in the previous section. Hence it is connected. To have a precise description of the topology of $\text{Stab}(P_n)$ we only have to understand how the Θ_k overlap. The answer is given by the following proposition.

Consider the abelian category \mathcal{Q}_n and set

$$\begin{aligned} Z^{-1}(S_0[1]) &= -1, & (\Rightarrow \phi(S_0[1]) &= 1), \\ Z^{-1}(S_1) &= 1 + i, & (\Rightarrow \phi(S_1) &= 1/4). \end{aligned}$$

By Remark 2.2, this extends to a unique stability condition $\sigma^{-1} = (Z^{-1}, \mathcal{P}^{-1})$ on $D(P_n)$. Consider its $\widetilde{GL^+(2, \mathbb{R})}$ -orbit O_{-1} , which is an open subset of $\text{Stab}(P_n)$ homeomorphic to $\widetilde{GL^+(2, \mathbb{R})}$. Notice that in the case $n = 2$, i.e. for \mathbb{P}^1 , the stability condition induced by μ -stability [8, Example 5.4] is in O_{-1} .

Proposition 4.4. *For all integers $k \neq h$ we have*

$$\Theta_k \cap \Theta_h = O_{-1}.$$

¹One should note the different uses of the term ‘‘orthogonal’’ here and in [19].

Proof. First of all, the fact that $O_{-1} \subseteq \Theta_k \cap \Theta_h$ is a simple consequence of Proposition 3.17. Let $\sigma \in \Theta_k \cap \Theta_h$. Set ϕ_0 and ϕ_1 the phases of $S_k[1]$ and S_{k+1} respectively. Then, since $\text{Hom}(S_k, S_{k+1}) \neq 0$, there exists an integer $p \geq -1$ such that

$$\phi_1 - 1 < \phi_0 + p \leq \phi_1.$$

But if $p \geq 0$ then there are no stable object in σ besides S_k and S_{k+1} and so $\sigma \notin \Theta_h$. Hence $p = -1$. If $\phi_0 = \phi_1 + 1$, then S_k and S_{k+1} are stable with the same phase, a contradiction. Hence $\phi_1 < \phi_0 < \phi_1 + 1$. We can then act by an element of $GL^+(2, \mathbb{R})$ and assuming $S_k[1], S_{k+1} \in \mathcal{P}((0, 1])$. By Lemma 3.16 $\langle S_k[1], S_{k+1} \rangle = \mathcal{P}((0, 1])$. So, either $S_0, S_1 \in \mathcal{P}((0, 1])$ or $S_0[1], S_1[1] \in \mathcal{P}((0, 1])$ and, by Proposition 3.17, they are both stable. But then acting again with an element of $GL^+(2, \mathbb{R})$ we can assume $S_0[1], S_1 \in \mathcal{P}((0, 1])$ and that the stability function coincides with Z_{-1} . But then again by Lemma 3.16 and by Remark 2.2, the resulting stability condition is σ^{-1} . ■

Theorem 4.5. *Stab(P_n) is a connected and simply connected 2-dimensional complex manifold.*

Proof. By Proposition 4.4, the simply connected open subsets Θ_k glue on O_{-1} , which is contractible. The theorem follows from the Seifert-Van Kampen Theorem. ■

Corollary 4.6. *Stab(\mathbb{P}^1) := Stab($D(\mathbb{P}^1)$) is connected and simply connected.*²

Note that the group $\text{Aut}(D(P_n))$ of autoequivalences of $D(P_n)$ acts transitively on the set $\{\Theta_k\}_{k \in \mathbb{Z}}$. Moreover, the subgroup of it which fixes Θ_k acts trivially on it, up to shifts. But the action of $\text{Aut}(D(P_n))$ on O_{-1} is nontrivial: it is an easy computation to see that

$$O_{-1} / \text{Aut}(D(P_n)) \cong GL^+(2, \mathbb{R}) / G,$$

where G is the subgroup generated by $\begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}$.

4.2. Stab(\mathbb{P}^N). We apply the results of the previous subsection to the study of $\text{Stab}(\mathbb{P}^N) := \text{Stab}(D(\mathbb{P}^N))$. By Example 3.12 we know that $D(\mathbb{P}^N)$ is generated by a strong exceptional collection consisting of sheaves. By a result of Bondal ([6, Theorem 9.3]) every complete exceptional sequence that consists of sheaves on \mathbb{P}^N is strong. Moreover, by [6, Assertion 9.2], a mutation of it consists of sheaves on \mathbb{P}^N (and so it is strong and there are no orthogonal exceptional pairs in it).

Let $\mathcal{E} = \{E_0, \dots, E_N\}$ be a strong complete exceptional collection on $D(\mathbb{P}^N)$ consisting of sheaves. As in the previous section we can define $\Theta_{\mathcal{E}}$ as the open connected and simply connected subset of $\text{Stab}(\mathbb{P}^N)$ consisting of stability conditions which are, up to the action of $GL^+(2, \mathbb{R})$, degenerate or non-degenerate for \mathcal{E} . Consider also the open connected subset $\Sigma_{\mathcal{E}}$ defined as the union of the open subsets $\Theta_{\mathcal{F}}$ over all iterated mutations \mathcal{F} of \mathcal{E} .

Theorem 4.7. *The closure of $\Theta_{\mathcal{E}}$ in $\text{Stab}(\mathbb{P}^N)$ is contained in $\Sigma_{\mathcal{E}}$.*

Proof. Let $\bar{\sigma} = (\bar{Z}, \bar{\mathcal{P}}) \in \partial\Theta_{\mathcal{E}}$. We already observed that E_1, \dots, E_N are stable in all stability conditions in $\Theta_{\mathcal{E}}$. Hence they are semistable in $\bar{\sigma}$ and there exist integers $p_0, \dots, p_N \in \mathbb{Z}$ and $l_1, \dots, l_h \in \{0, \dots, N\}$, $l_1 < \dots < l_h$ such that, up to the action of $GL^+(2, \mathbb{R})$,

$$\mathcal{E}' = \{E_0[p_0], \dots, E_{l_1}[p_{l_1} + 1], E_{l_1+1}[p_{l_1+1}], \dots, E_{l_h}[p_{l_h} + 1], \dots, E_N[p_N]\} \subseteq \bar{\mathcal{P}}((0, 1])$$

and the exceptional collection $\{E_0[p_0], \dots, E_i[p_i], \dots, E_N[p_N]\}$ is Ext. Since $\bar{\sigma}$ is in the boundary of $\Theta_{\mathcal{E}}$, the exceptional collection \mathcal{E}' is not Ext. Assume that \mathcal{E}' has the form

$$\{E_0[p_0], \dots, E_{i-1}[p_{i-1}], E_i[p_i + 1], E_{i+1}[p_{i+1}], \dots, E_N[p_N]\},$$

with $0 = \bar{\phi}(E_i[p_i]) = \bar{\phi}(E_j[p_j]) - 1$ for only one j . Since $\Sigma_{\mathcal{E}}$ is locally euclidean and $\bar{\sigma}$ is locally finite, the general case can be reduced to this case by induction. Let σ_s be a sequence of stability conditions in $\Theta_{\mathcal{E}}$ such that $\sigma_s \rightarrow \bar{\sigma}$. We can assume further that σ_s belongs, for all s , to the open subset U_p , defined in the proof of Lemma 3.18. Then $\phi_s(E_i[p_i]) - \phi_s(E_j[p_j]) + 1 \rightarrow 0$, where

²The same result has been obtained independently by Okada [18]. Actually in that paper it is proved a stronger statement: $\text{Stab}(\mathbb{P}^1) \cong \mathbb{C}^2$.

$\phi_s(E)$ is the phase of the semistable object E in σ_s . So, for $s \gg 0$, $\phi_s(E_i[p_i]) < \phi_s(E_l[p_l])$, $\forall l \neq i$ (note that, for $i = 0$, \mathcal{E}' is an Ext-exceptional collection). Let E_{i_1}, \dots, E_{i_k} be such that $\text{Hom}(E_{i_r}[p_{i_r}], E_i[p_i+1]) \neq 0$, $r = 1, \dots, k$. Since the collection $\{E_0, \dots, E_N\}$ is strong exceptional and there are no orthogonal exceptional pairs in it, we have $i_1 = \dots = i_k = i - 1$.

Assume $j = i - 1$. By Proposition 3.17 σ_s induces a stability condition on $\text{Tr}(E_{i-1}, E_i)$ ($\cong D^b(P_m)$, with $m = \dim \text{Ext}^1(E_{i-1}[p_{i-1}], E_i[p_i])$) and this induced stability condition lies in the open subset O_{-1} of $\text{Stab}(P_m)$. This implies that $\bar{\sigma}$ induces a stability condition on $D^b(P_m)$. By Lemma 4.2 there exists an exceptional pair (F_{i-1}, F_i) consisting of shifts of an iterated mutation of (E_{i-1}, E_i) such that $\text{Hom}(F_{i-1}, F_i) = 0$ and, up to the action of $\widetilde{GL^+(2, \mathbb{R})}$, the exceptional collection

$$\mathcal{E}'' = \{E_0[p_0], \dots, E_{i-2}[p_{i-2}], F_{i-1}, F_i, E_{i+1}[p_{i+1}], \dots, E_N[p_N]\},$$

is an Ext-exceptional collection consisting of shifts of sheaves in $\overline{P}((0, 1])$. So, $\bar{\sigma} \in \Theta_{\mathcal{E}''}$, by Lemma 3.16.

If $j \neq i - 1$, then $\bar{\sigma} \in \Theta_{\mathcal{E}'''}$, where

$$\mathcal{E}''' = \{E_0, \dots, E_j, \dots, E_{i-2}, E_i[1], \mathcal{R}_{E_i} E_{i-1}[-1], E_{i+1}, \dots, E_N\}.$$

■

Remark 4.8. Notice that in the stability condition $\bar{\sigma} \in \Theta_{\mathcal{E}'''}$ of the last part of the proof of the previous theorem, all the objects E_0, \dots, E_N are stable but $\bar{\sigma} \notin \Theta_{\mathcal{E}}$.

In the case $N = 2$ we can say a little bit more about the topology of this open subset. First of all $D(\mathbb{P}^2)$ is constructible [15] (in particular, all complete exceptional collections consist of sheaves, up to shifts) and so the action of the braid group A_3 on the set of exceptional collections consisting of sheaves is transitive. Secondly this action is also free (see e.g. [10, Theorem 5.5]).

To begin with, we rewrite Lemma 3.18 as

Lemma 4.9. *Let $\{E_0, E_1, E_2\}$ be a complete exceptional collection on $D(\mathbb{P}^2)$ consisting of sheaves. Then $\Theta_{\mathcal{E}}$ is homeomorphic to*

$$\mathcal{C}_{\mathcal{E}} = \left\{ (m_0, m_1, m_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \begin{array}{l} \bullet m_i > 0, \text{ for all } i \\ \bullet \phi_0 < \phi_1 < \phi_2, \phi_0 < \phi_2 - 1 \end{array} \right\}.$$

Then we study intersections of the open subsets Θ .

Lemma 4.10. *Let $\mathcal{E} = \{E_0, \dots, E_n\}$ and $\mathcal{F} = \{F_0, \dots, F_n\}$ be complete exceptional collections on a triangulated category \mathbf{T} . Assume $\Theta_{\mathcal{E}} \cap \Theta_{\mathcal{F}} \neq \emptyset$. Then either $\Theta_{\mathcal{E}} = \Theta_{\mathcal{F}}$ or there exists a degenerate stability condition $\bar{\sigma} \in \partial\Theta_{\mathcal{E}} \cap \Theta_{\mathcal{F}}$.*

Proof. By hypothesis, either $\Theta_{\mathcal{E}} = \Theta_{\mathcal{F}}$ or there exists a stability condition $\sigma \in \partial\Theta_{\mathcal{E}} \cap \Theta_{\mathcal{F}}$. Now, for every stability condition in $\Theta_{\mathcal{F}}$ there exists a sequence $G_i \in \widetilde{GL^+(2, \mathbb{R})}$ such that $G_i \sigma \rightarrow \bar{\sigma}$, where $\bar{\sigma}$ is a degenerate stability condition for \mathcal{F} . But then $\bar{\sigma} \in \partial\Theta_{\mathcal{E}} \cap \Theta_{\mathcal{F}}$. ■

Finally we study more explicitly the boundary.

Lemma 4.11. *Let $\{E_0, E_1, E_2\}$ and $\mathcal{F} = \{F_0, F_1, F_2\}$ be complete exceptional collections on $D(\mathbb{P}^2)$ consisting of sheaves. If $\bar{\sigma} \in \partial\Theta_{\mathcal{E}} \cap \Theta_{\mathcal{F}}$ is degenerate, then there exists $l = l_s \dots l_1 \in A_3$, $l_k \in \{\mathcal{L}_0, \mathcal{L}_1, \mathcal{R}_0, \mathcal{R}_1\}$ for all $k \in \{1, \dots, s\}$, such that $\mathcal{F} = l\mathcal{E}$ and there exist real numbers $a_0 = 0 < a_1 < \dots < a_s < a_{s+1} = 1$ and a continuous path $\tau : [0, 1] \rightarrow \Sigma(\mathbb{P}^2)$ such that $\tau([a_k, a_{k+1})) \subseteq \Theta_{(l_k \dots l_1 \mathcal{E})} \cap \Theta_{\mathcal{E}}$ and $\tau(1) = \bar{\sigma}$.*

Proof. It follows immediately from the first part of the proof of Theorem 4.7. ■

Corollary 4.12. $\Sigma(\mathbb{P}^2)$ is a 3-dimensional connected and simply connected manifold.

Proof. Take a continuous loop $\tau : [0, 1] \rightarrow \Sigma(\mathbb{P}^2)$ with base point x_0 in $\Theta_0 = \Theta_{\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)}$. Then, by Lemma 4.11, there exist real numbers $a_0 = 0 < a_1 < \dots < a_m < a_{m+1} = 1$, $m \in \mathbb{N}$, and a family of complete exceptional collections consisting of sheaves

$$\{\mathcal{M}_k = \{M_0^k, M_1^k, M_2^k\}\}_{k \in \{0, \dots, m\}}$$

with $\mathcal{M}_0 = \{\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)\}$ and \mathcal{M}_{k+1} obtained from \mathcal{M}_k by a single mutation γ_{k+1} such that $\tau([a_k, a_{k+1}]) \subseteq \Theta_k := \Theta_{\mathcal{M}_k}$ for $k \in \{0, \dots, m\}$ and $x_0 \in \Theta_m \cap \Theta_0$. By Lemma 4.10, Lemma 4.9 and again Lemma 4.11, we can assume $\Theta_m = \Theta_0$. So, $\gamma = \gamma_m \dots \gamma_1$ has the property that

$$\gamma(\{\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)\}) = \{\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)\}.$$

Since the braid group A_3 acts freely on the set of complete exceptional collections on \mathbb{P}^2 , then γ must be the identity in A_3 . Now, if $\gamma = \text{id}_{A_3}$ in A_3 , then, up to contracting/adding pieces of the form ll^{-1} , for $l \in L$, where L is the free group generated by $\mathcal{L}_0, \mathcal{L}_1$, we have

$$\gamma = (h_1 r^{\pm 1} h_1^{-1}) \dots (h_s r^{\pm 1} h_s^{-1}),$$

with $r = \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_1 \mathcal{L}_2 \mathcal{L}_1 \mathcal{L}_2$, and $h_1, \dots, h_s \in L$. But an explicit computation, using Lemma 4.9, shows that if $\gamma = \mathcal{R}_0 \mathcal{R}_1 \mathcal{R}_0 \mathcal{L}_1 \mathcal{L}_0 \mathcal{L}_1$ (or its inverse), then τ can be contracted in $\Sigma(\mathbb{P}^2)$. Moreover, if $\gamma = ll^{-1}$, $l \in L$, then again τ can be contracted. But this implies that τ can be contracted in general, i.e. that $\Sigma(\mathbb{P}^2)$ is simply-connected. \blacksquare

It seems reasonable to us, but unfortunately we are not able to give a rigorous proof, that the open subset $\Sigma(\mathbb{P}^2)$ can be completed to a whole connected component of $\text{Stab}(\mathbb{P}^2)$, by adding “geometric” stability conditions, constructed along the line of [9, Section 6]. The previous results can be seen as a first step on this direction.

4.3. Del Pezzo surfaces and other generalizations. The results of the previous subsection can be easily generalized to derived categories generated by strong exceptional collections whose iterated mutations are again strong. For example, by [6], derived categories of smooth projective varieties Z that have complete exceptional collections consisting of sheaves and that satisfy

$$\text{rk } K(Z) = \dim Z + 1.^3$$

Here we study the simplest case in which this condition is not verified: Del Pezzo surfaces.

Let S be a Del Pezzo surface, $n = \text{rk } K(S) = \text{rk } H^*(S, \mathbb{Z})$. Exceptional collections on Del Pezzo surfaces were studied exhaustively in [17]. In particular, one can prove that every exceptional collection is regular, consists of shift of sheaves and that $D(S)$ is constructible. The only difference with respect to \mathbb{P}^N is that it is not true that, also up to shifts, every mutation of a strong exceptional collection is again strong exceptional. But this is not really important for our study.

Theorem 4.13. *Let $\mathcal{E} = \{E_0, \dots, E_n\}$ be a complete exceptional collection on S consisting of sheaves. Then the closure of $\Theta_{\mathcal{E}}$ in $\text{Stab}(S)$ is contained in Σ_S .*

Proof. The proof goes along the same lines as Theorem 4.7. Let $\bar{\sigma} = (\bar{Z}, \bar{\mathcal{P}}) \in \partial\Theta_{\mathcal{E}}$. Proceeding as in the proof of Theorem 4.7 we have that E_0, \dots, E_n are semistable in $\bar{\sigma}$ and we can reduce to the situation in which

$$\mathcal{E}' = \{E_0[p_0], \dots, E_{i-1}[p_{i-1}], E_i[p_i + 1], E_{i+1}[p_{i+1}], \dots, E_n[p_n]\} \subseteq \bar{\mathcal{P}}((0, 1)),$$

with $0 = \bar{\phi}(E_i[p_i]) = \bar{\phi}(E_j[p_j]) - 1$ for only one j , where the exceptional collection \mathcal{E}' is not Ext and $\{E_0[p_0], \dots, E_i[p_i], \dots, E_n[p_n]\}$ is Ext. Consider the exceptional pair (E_j, E_i) .

i) Assume $\text{Hom}(E_j[p_j], E_i[p_i + 1]) \neq 0$. Then, by Proposition 3.17, $\bar{\sigma}$ induces a stability condition on $\text{Tr}(E_j, E_i) \cong D^b(P_m)$, for $m = \dim \text{Hom}(E_j[p_j], E_i[p_i + 1])$. By Lemma 4.2 there exists an exceptional pair (N_i, N_j) consisting of shifts of an iterated mutation of (E_j, E_i) such that $\text{Hom}^{\leq 0}(N_i, N_j) = 0$ and, up to the action of $\widetilde{GL^+(2, \mathbb{R})}$,

$$\{N_i, N_j\} \subseteq \bar{\mathcal{P}}((0, 1)).$$

If (N_i, N_j) is an iterated left mutation of (E_j, E_i) then we have to consider the case

$$\mathcal{E}' = \{E_0[p_0], \dots, E_{j-1}[p_{j-1}], E_j[p_j - 1], E_{j+1}[p_{j+1}], \dots, E_i[p_i], \dots, E_n[p_n]\} \subseteq \bar{\mathcal{P}}([0, 1]),$$

which up to the action of $\widetilde{GL^+(2, \mathbb{R})}$ is equivalent to our case and can be dealt with by a similar procedure of that we will see (changing right and left mutations).

³See also [10].

So, assume that (N_i, N_j) is an iterated right mutation of (E_j, E_i) . By [17], given an exceptional pair consisting of locally-free sheaves on S , it is easy to determine if it is an Ext pair (i.e. the first Ext group is different from zero) or an Hom pair (i.e. there are non-trivial homomorphisms) just looking at degree. Using this and the fact ([17, Proposition 2.9]) that an exceptional sheaf on S is either locally-free or torsion supported on a (-1) -curve, it is straightforward to see that, up to switching orthogonal pairs of exceptional objects, for all integers k such that $j < k < i$, either $\text{Hom}(E_k[p_k], E_i[p_i + 1]) \neq 0$ or (E_k, E_i) orthogonal exceptional pair. Unlike the case of projective spaces, in general it is not true anymore that $j = i - 1$. To find an Ext exceptional collection \mathcal{F} such that $\mathcal{F} \subseteq \overline{\mathcal{P}}((0, 1])$, we have to proceed in a slightly different way.

Assume $j < i - 1$. Then, if (E_{i-1}, E_i) is an orthogonal pair, we set $M_{i-1} := E_{i-1}[p_{i-1}]$ and we switch E_{i-1} and E_i . Otherwise, if $\text{Hom}(E_{i-1}[p_{i-1}], E_i[p_i]) \neq 0$, we define M_{i-1} as the appropriate shift of the right mutation $\mathcal{R}_{E_i} E_{i-1}$ such that $M_{i-1} \in \langle E_{i-1}[p_{i-1}], E_i[p_i] \rangle$. In both the two cases we get

$$\{\dots, E_{i-2}[p_{i-2}], E_i[p_i + 1], M_{i-1}, E_{i+1}[p_{i+1}], \dots\} \subseteq \overline{\mathcal{P}}((0, 1]).$$

Proceeding in this way, we get a sequence of $\bar{\sigma}$ -stable exceptional objects M_{j+1}, \dots, M_{i-1} such that

$$\{\dots, E_j[p_j], E_i[p_i + 1], M_{j+1}, \dots, M_{i-1}, E_{i+1}[p_{i+1}], \dots\} \subseteq \overline{\mathcal{P}}((0, 1])$$

and $\text{Hom}^{\leq 0}(M_\gamma, E_l[p_l]) = 0$, for all $l \neq \gamma$, $\text{Hom}^{\neq 0}(E_i[p_i], M_\gamma) = 0$, $\text{Hom}^{\leq 0}(E_{l'}[p_{l'}], M_\gamma) = 0$, for all $l' \neq i$.

Now we mutate (E_j, E_i) to (N_i, N_j) and we have

$$\{\dots, E_{j-1}[p_{j-1}], N_i, N_j, M_{j+1}, \dots, M_{i-1}, E_{i+1}[p_{i+1}], \dots\} \subseteq \overline{\mathcal{P}}((0, 1]),$$

with $N_i \in \langle E_j[p_j], E_i[p_i] \rangle[1]$ and $N_j \in \langle E_j[p_j], E_i[p_i] \rangle$.

This is ‘‘almost’’ an Ext exceptional collection. Indeed the only problem is that it may happen that $\text{Hom}(E_q[p_q], N_i) \neq 0$, for $q < j$. Set q_0 the minimum of such integers q . As before, up to switching orthogonal objects, for all $q_0 \leq q < j$, either $\text{Hom}(E_q[p_q], N_i) \neq 0$ or (E_q, N_i) is an orthogonal pair. In the second case, we set $R_q := E_q[p_q]$ and we switch E_q and N_i . In the first case, if $\text{Hom}(E_q[p_q], N_i) \neq 0$, we define R_q as the appropriate shift of the right mutation $\mathcal{R}_{N_i} E_q$ such that $R_q \in \langle E_q[p_q], N_i[-1] \rangle$. In both the two cases we get

$$\{\dots, E_{q_0-1}[p_{q_0-1}], N_i, R_{q_0}, \dots, R_{j-1}, N_j, M_{j+1}, \dots, M_{i-1}, E_{i+1}[p_{i+1}], \dots\} \subseteq \overline{\mathcal{P}}((0, 1]).$$

Note that it is not a priori obvious that R_q belongs to $\overline{\mathcal{P}}((0, 1])$. But this is an easy consequence of the fact that, if R_q is not semistable, then no element of its HN filtration can be of phase zero.

Finally, it may happen that $\text{Hom}(R_t, N_j) \neq 0$, for some t such that $q_0 \leq t < j$. Let t_0 be the minimum of such integers t . Proceeding precisely in the same way as before, we end with an exceptional collection

$$\mathcal{E}'' = \{\dots, E_{q_0-1}[p_{q_0-1}], N_i, R_{q_0}, \dots, R_{t_0-1}, N_j, S_{t_0}, \dots, S_{j-1}, M_{j+1}, \dots, M_{i-1}, E_{i+1}[p_{i+1}], \dots\}$$

contained in $\overline{\mathcal{P}}((0, 1])$, where S_t is the appropriate right mutation of the pair (R_t, N_j) . By construction and by Lemma 4.14 the exceptional collection \mathcal{E}'' is Ext. Hence, by Lemma 3.16, $\bar{\sigma} \in \Theta_{\mathcal{E}''}$.

ii) If $\text{Hom}(E_j[p_j], E_i[p_i + 1]) = 0$, then $\bar{\sigma} \in \Theta_{\mathcal{E}'''}$, where, for some integer k ,

$$\mathcal{E}''' = \{E_0[p_0], \dots, E_j[p_j], \dots, E_{i-k-1}[p_{i-k-1}], E_i[p_i + 1], M_{i-k}, \dots, M_{i-1}, E_{i+1}[p_{i+1}], \dots, E_n[p_n]\}.$$

■

Lemma 4.14. *Consider an Ext exceptional triple $\{E_q[p_q], E_j[p_j], E_i[p_i]\}$ on $D(S)$, with E_q, E_j, E_i sheaves. Assume $\text{Hom}(E_j[p_j], E_i[p_i + 1]) \neq 0$. Let (N_i, N_j) be a shift of $\mathcal{R}^a(E_j, E_i)$, $a > 0$ an integer, such that $N_i \in \langle E_j[p_j], E_i[p_i] \rangle[1]$ and $N_j \in \langle E_j[p_j], E_i[p_i] \rangle$. Assume $\text{Hom}(E_q[p_q], N_i) \neq 0$ and denote by R_q the shift of $\mathcal{R}_{N_i} E_q$ such that $R_q \in \langle E_q[p_q], N_i[-1] \rangle$. If $\text{Hom}(R_q, N_j) \neq 0$, then $\text{Hom}(E_q[p_q], E_j[p_j + 1]) \neq 0$ and, if S_q denote the shift of $\mathcal{R}_{N_j} R_q$ such that $S_q \in \langle R_q, N_j[-1] \rangle$, then $S_q \in \langle E_q[p_q], E_j[p_j], E_i[p_i] \rangle$.*

Proof. It is a straightforward computation to show that $\text{Hom}(E_q[p_q], E_j[p_j + 1]) \neq 0$. Now, we use Proposition 3.9, (iii), which can be rewritten as

$$\mathcal{R}_1 \mathcal{R}_0 \mathcal{R}_1^a = \mathcal{R}_0^a \mathcal{R}_1 \mathcal{R}_0. \quad (7)$$

Let Q_q be the shift of $\mathcal{R}_{E_j} E_q$ such that $Q_q \in \langle E_q[p_q], E_j[p_j] \rangle$. Then $\text{Hom}(Q_q, E_i[p_i + 1]) \neq 0$. Denote by T_q the shift of $\mathcal{R}_{E_i} Q_q$ such that $T_q \in \langle Q_q, E_i[p_i] \rangle = \langle E_q[p_q], E_j[p_j], E_i[p_i] \rangle$. Then, by (7), $S_q = T_q \in \langle E_q[p_q], E_j[p_j], E_i[p_i] \rangle$. ■

APPENDIX A. STABILITY CONDITIONS ON CURVES OF POSITIVE GENUS

Let C be a smooth projective curve over \mathbb{C} of positive genus. The Grothendieck group of $D(C) := D^b(\mathbf{Coh}(C))$ is not anymore of finite rank as in the examples seen in the rest of the paper. Hence the stability manifold $\text{Stab}(D(C))$ is infinite dimensional. We can restrict to a finite dimensional (more geometric) slice of it. According to [8], we define $\text{Stab}(C)$ as the finite dimensional submanifold of $\text{Stab}(D(C))$ consisting of locally finite *numerical stability conditions*, i.e. locally finite stability conditions whose central charge factorizes through the singular cohomology $H^*(C, \mathbb{Z})$ of C , via the Chern character. An example of numerical stability condition on the bounded derived category of a curve is the stability condition induced by μ -stability for sheaves [8, Example 5.4].

The fundamental ingredient for studying $\text{Stab}(C)$ is this technical lemma [14, Lemma 7.2].

Lemma A.1. *Let C a smooth projective curve of genus $g(C) \geq 1$. Suppose $E \in \mathbf{Coh}(C)$ is included in a triangle*

$$Y \rightarrow E \rightarrow X \rightarrow Y[1]$$

with $\text{Hom}^{\leq 0}(Y, X) = 0$. Then $X, Y \in \mathbf{Coh}(C)$.

Theorem A.2. *If C has genus $g(C) \geq 1$, then the action of $\widetilde{GL}^+(2, \mathbb{R})$ on $\text{Stab}(C)$ is free and transitive, so that*

$$\text{Stab}(C) \cong \widetilde{GL}^+(2, \mathbb{R}).$$

Proof. First note that the structure sheaves of points are stable. Indeed they are semistable because otherwise, by Lemma A.1, \mathcal{O}_x is included in an exact sequence in $\mathbf{Coh}(C)$

$$0 \rightarrow Y \rightarrow \mathcal{O}_x \rightarrow X \rightarrow 0,$$

with $\text{Hom}^{\leq 0}(Y, X) = 0$, which is clearly impossible. Now, if \mathcal{O}_x were not stable, then by the same argument all its stable factors should be isomorphic to a single object K , which implies $K \in \mathbf{Coh}(C)$ and so $K \cong \mathcal{O}_x$. In the same way, all line bundles are stable too.

Then, let $\sigma = (Z, \mathcal{P}) \in \text{Stab}(C)$. Take a line bundle A on C ; by what we have seen above, A and \mathcal{O}_x are stable in σ with phases ϕ_A and ψ respectively. The existence of maps $A \rightarrow \mathcal{O}_x$ and $\mathcal{O}_x \rightarrow A[1]$ gives inequalities $\psi - 1 \leq \phi_A \leq \psi$, which implies that if Z is an isomorphism (seen as a map from $H^*(C, \mathbb{R}) \cong \mathbb{R}^2$ to $\mathbb{C} \cong \mathbb{R}^2$) then it must be orientation preserving. But Z is an isomorphism: indeed if not, then there exist stable objects with the same phase having non-trivial morphisms, which is impossible. Hence, acting by an element of $\widetilde{GL}^+(2, \mathbb{R})$, one can assume that $Z(E) = -\text{deg}(E) + i \text{rk}(E)$ and that for some $x \in C$, the skyscraper sheaf \mathcal{O}_x has phase 1. Then all line bundles on C are stable in σ with phases in the interval $(0, 1)$ and all structure sheaves of points are stable of phase 1; but this implies that $\mathcal{P}((0, 1]) = \mathbf{Coh}(C)$ and so that the stability condition σ is precisely the one induced by μ -stability on C . ■

REFERENCES

- [1] P.S. Aspinwall and I.V. Melnikov, *D-branes on vanishing Del Pezzo surfaces*. J. High Energy Phys. 2004, no. 12, 042, 30 pp. (2005), also hep-th/0405134.
- [2] P.S. Aspinwall, *D-Branes, Pi-Stability and Theta-Stability*, preprint hep-th/0407123.
- [3] A.A. Beilinson, *Coherent sheaves on \mathbb{P}^n and problems in linear algebra*. Funktsional. Anal. i Prilozhen. 12 (1978), no. 3, 68–69; English transl. in Functional Anal. Appl. 12 (1978), no.3, 214–216.
- [4] A.A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux Pervers*. Astérisque 100, Soc. Math. de France (1982).
- [5] D.J. Benson, *Representations and Cohomology I*. Camb. Stud. in Adv. Math., vol.30, C.U.P. (1991).
- [6] A.I. Bondal, *Representations of associative algebras and coherent sheaves*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 1, 25–44; translation in Math. USSR-Izv. 34 (1990), no. 1, 23–42.
- [7] A.I. Bondal and M.M. Kapranov, *Representable functors, Serre functors, and reconstructions*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 1183–1205, 1337; translation in Math. USSR-Izv. 35 (1990), no. 3, 519–541.

- [8] T. Bridgeland, *Stability conditions on triangulated categories*, preprint math.AG/0212237. To appear in Ann. Math.
- [9] T. Bridgeland, *Stability conditions on K3 surfaces*, preprint math.AG/0307164.
- [10] T. Bridgeland, *T-structures on some local Calabi-Yau varieties*. J. Algebra 289 (2005), no. 2, 453–483, also math.AG/0502050.
- [11] W. Crawley-Boevey, *Exceptional sequences of representations of quivers*. Representations of algebras (Ottawa, ON, 1992), 117–124, CMS Conf. Proc., 14, Amer. Math. Soc., Providence, RI (1993).
- [12] M.R. Douglas, *Dirichlet branes, homological mirror symmetry, and stability*. Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 395–408, Higher Ed. Press (2002), also math.AG/0207021.
- [13] S.I. Gelfand and Y.I. Manin, *Methods of homological algebra*. Translated from the 1988 Russian original. Springer-Verlag, Berlin (1996).
- [14] A.L. Gorodentsev, S.A. Kuleshov and A.N. Rudakov, *t-stabilities and t-structures on triangulated categories*. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 68 (2004), no. 4, 117–150; translation in Izv. Math. 68 (2004), no. 4, 749–781, also math.AG/0312442.
- [15] A.L. Gorodentsev and A.N. Rudakov, *Exceptional vector bundles on projective spaces*. Duke Math. J. 54 (1987), no. 1, 115–130.
- [16] A.D. King, *Moduli of representations of finite-dimensional algebras*. Quart. J. Math. Oxford Ser. (2) 45 (1994), no. 180, 515–530.
- [17] S.A. Kuleshov and D.O. Orlov, *Exceptional sheaves on Del Pezzo surfaces*. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 58 (1994), no. 3, 53–87; translation in Russian Acad. Sci. Izv. Math. 44 (1995), no. 3, 479–513.
- [18] S. Okada, *Stability manifold of \mathbb{P}^1* . J. Algebraic Geom. 15 (2006), no. 3, 487–505, also math.AG/0411220.
- [19] C.M. Ringel, *The braid group action on the set of exceptional sequences of a hereditary Artin algebra*. Abelian group theory and related topics (Oberwolfach, 1993), 339–352, Contemp. Math., 171, Amer. Math. Soc., Providence, RI (1994).

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