Midterm exam 2 - Calculus III, Math 2210, Sect. 5
April 2, 2009

Rules.

- The exam is **closed books** and **no calculators** and consists of **four** exercises.
- To write the solution use the space on the sheet below the statement of the exercise and in the next page. If you need further paper, ask me. Do not use your paper.
- Justify **ALL** your answers.
- The first exercise counts 4 points. The second and the third 3 points each. The last one 2 points.
- The total grade of the exam will be **10 points** (plus 2 extra points). Hence you have to do as much as you can of the four exercises to reach 10 points.

Exercise 1. Let $f(x,y) = \frac{xy}{y^2+1} + \frac{xy}{x^4+1}$. Evaluate

$$\int_S f(x,y)\,dxdy,$$

where S is the right triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$ (see Figure 1).
(**Recall:**

$$\int \ln(u)\,du = u\ln(u) - u.)$$

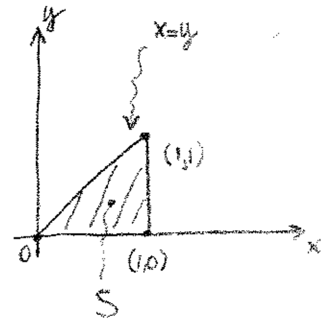

Figure 1.

$$S = \left\{ (x,y) : \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq x \end{array} \right\}$$

$$\rightsquigarrow \int_S f(x,y)\,dxdy =$$

$$= \underbrace{\int_0^1 \int_0^x x \cdot \frac{y}{y^2+1}\,dy\,dx} + \int_0^1 \int_0^x \frac{x}{x^4+1}\, y\,dy\,dx =$$

$$\int_0^1 x \left[\ln(y^2+1) \cdot \frac{1}{2} \right]_0^x dx =$$

$$= \frac{1}{2} \int_0^1 x \ln(x^2+1)\,dx = \frac{1}{2} \cdot \frac{1}{2} \left[(x^2+1)\ln(x^2+1) - (x^2+1) \right]_0^1$$

$$= \frac{1}{4} \left(2\ln(2) - 2 - 0 + 1 \right) =$$

$$= \boxed{\frac{\ln(2)}{2} - \frac{1}{4}}$$

For the second part,

$$\int_0^1 \int_0^x \frac{x}{x^4+1} \cdot y \, dy \, dx = \int_0^1 \frac{x}{x^4+1} \cdot \left[\frac{y^2}{2} \right]_0^x dx =$$

$$= \frac{1}{2} \int_0^1 \frac{x^3}{x^4+1} \, dx = \frac{1}{2} \left[\ln(x^4+1) \cdot \frac{1}{4} \right]_0^1 =$$

$$= \boxed{\frac{1}{8} \ln 2}$$

$$\rightsquigarrow \int_S f(x,y) \, dx \, dy = \frac{\ln(2)}{2} - \frac{1}{4} + \frac{\ln(2)}{8} = \boxed{\frac{5}{8} \ln(2) - \frac{1}{4}}$$

Exercise 2. Using double integrals, compute the area of the surface in 3D with equation $z^2 = x^2 + y^2$ above the plane $z = -1$ and below the plane $z = 1$ (see Figure 2).

<u>Sol</u>: We have, by symmetry,

$$\text{Area} = 2 \int_S \sqrt{\partial_x f^2 + \partial_y f^2 + 1}\, dx\,dy,$$

where $f(x,y) = \sqrt{x^2+y^2}$

and $S = \{(x,y) : x^2+y^2 \leq 1\}$.

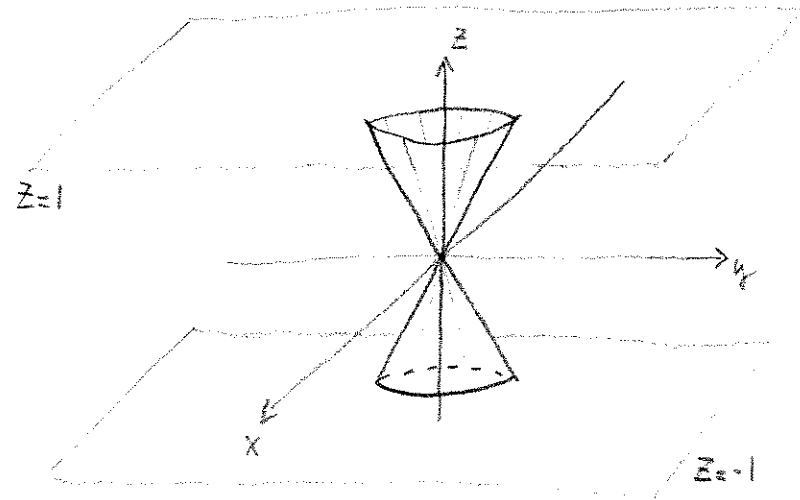

Figure 2.

$$\partial_x f = \frac{x}{\sqrt{x^2+y^2}}$$

$$\partial_y f = \frac{y}{\sqrt{x^2+y^2}}$$

$$\rightsquigarrow \int_S \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}\, dx\,dy =$$

$$= \int_S \sqrt{\frac{2(x^2+y^2)}{x^2+y^2}}\, dx\,dy = \int_S \sqrt{2}\, dx\,dy = \pi\sqrt{2}$$

$$\rightsquigarrow \text{Area} = \boxed{2\sqrt{2}\,\pi}.$$

Exercise 3. Evaluate
$$\int_S \frac{1}{x^2+y^2}\,dxdy,$$
where S is the region in the plane
$$S := \{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 4\}$$
(see Figure 3).

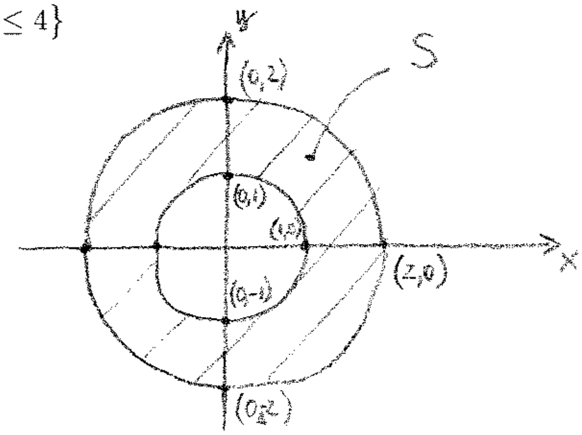

Figure 3.

Sol: We use polar coordinates.

$$S = \left\{ (r,\theta) : \begin{array}{c} 1 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

$$\rightsquigarrow \int_S \frac{1}{x^2+y^2}\,dxdy = \int_0^{2\pi}\int_1^2 \frac{1}{r^2}\cdot r\,dr\,d\theta$$

$$= 2\pi \int_1^2 \frac{1}{r}\,dr = 2\pi \left[\ln r\right]_1^2 = \boxed{2\pi \ln 2}.$$

Exercise 4. Show that
$$\int_R x e^{x^{100}+y}\, dxdy = 0,$$
where R is the rectangle
$$R := \left\{(x,y) \in \mathbb{R}^2 \,:\, -1 \le x \le 1,\, 0 \le y \le 1\right\}.$$
(**Suggestion:** Do not try to solve it explicitly! Look at the function, the domain of integration, and use symmetry...)

Sol: $R = R_1 \cup R_2$, where

$$R_1 = \left\{(x,y) : \begin{array}{l} -1 \le x \le 0 \\ 0 \le y \le 1 \end{array}\right\}$$

$$R_2 = \left\{(x,y) : \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array}\right\}.$$

Moreover, $f(x,y) = x e^{x^{100}+y}$

satisfies $f(-x,y) = -x e^{x^{100}+y} = -f(x,y)$

Hence $$\int_{R_1} f(x,y)\,dxdy = -\int_{R_2} f(x,y)\,dxdy$$

$$\leadsto \int_R f(x,y)\,dxdy = \int_{R_1} f(x,y)\,dxdy + \int_{R_2} f(x,y)\,dxdy = 0$$