First midterm exam - Calculus for Engineers, Math 1280, Sect. 1
February 23, 2010

Rules.
- The exam is closed books and no calculators and consists of two exercises.
- To write the solution use the space on the sheet below the statement of the exercise and in the next page. If you need further paper, ask me. Do not use your paper.
- Justify ALL your answers.
- The total grade of the exam will be 10 points (plus 2 extra points). Hence you have to do as much as you can of the two exercises to reach 10 points.
Then we rotate:

\[ \text{(7) No} \]

Indeed, if \( x = 0 \), we have

\[
\lim_{t \to 0} \cos \left( \frac{1}{t^2} \right) \text{ which does not exist.}
\]
Exercise 1. Let $f(x, y) = \cos \left( \frac{1}{x^2 + y^2} \right)$ be a function of two variables.

(1) (1 point) What is the domain $\text{Dom}(f)$ of $f$?
(2) (2 points) What is the range of $f$?
(3) (1 point) Is $f$ continuous on $\text{Dom}(f)$?
(4) (1 point) Compute $\partial_x f(x, y)$ and $\partial_y f(x, y)$.
(5) (1 point) Find the tangent plane to $z = f(x, y)$ at $(x_0, y_0) = (0, 1)$.
(6) (2 points) Sketch the graph $z = f(x, y)$.
(7) (facultative, 2 points) Does

$$\lim_{(x, y) \to (0, 0)} f(x, y)$$

exist? Why?

\[ \text{Sol} : \]

(1) $\text{Dom}(f) = \mathbb{R}^2 - \{ (0, 0) \}$

(2) $\text{Range} = [-1, 1] \Leftarrow$ If $\frac{1}{x^2 + y^2} = 0$, then $\cos(\frac{1}{x^2 + y^2}) = 1$

(3) $x^2 + y^2$ polynomial $\Rightarrow$ continuous

$\frac{1}{x}$ continuous

$\cos(t)$ continuous

$\Rightarrow f(x, y) = \cos \left( \frac{1}{x^2 + y^2} \right)$ continuous

(4) $\partial_x f(x, y) = -\sin \left( \frac{1}{x^2 + y^2} \right) \cdot \left( -\frac{1}{(x^2 + y^2)^2} \right) \cdot 2x$

$$= 2x \frac{1}{(x^2 + y^2)^2} \cdot \sin \left( \frac{1}{x^2 + y^2} \right)$$
\[ \partial_y \tilde{f}(x, y) = 2 \frac{y}{(x^2 + y^2)^2} \sin \left( \frac{1}{x^2 + y^2} \right) \]

(5) \( T_0 \) plane

\[ Z = f(0, 1) + \partial_x f(0, 1) \cdot x + \partial_y f(0, 1) \cdot (y - 1) \]

\[ \Rightarrow Z = \cos(1) + 2 \sin(1)(y - 1) \]

\[ \Rightarrow Z = 2 \sin(1) y + \cos(1) - 2 \sin(1) \]

(6) The graph \( Z = \cos \left( \frac{1}{x^2 + y^2} \right) \) is symmetric in \( x^2 + y^2 \)

\[ \Rightarrow \text{obtained by rotation along } \underline{z}-\text{axis} \]

First, intersect with \( x = 0 \) plane \( \Rightarrow Z = \cos \left( \frac{1}{y^2} \right) \)

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Exercise 2. Consider the two lines in 2D (with coordinates \((x, y)\))

\[ l_1 : \ 2\lambda x + \lambda y = 1, \]

where \(\lambda\) is a real number, and

\[ l_2 : \ 2x - y = 0. \]

1. (1 point) For which values of \(\lambda\) the two lines intersect in exactly one point?
2. (1 point) For which values of \(\lambda\) the two lines are parallel? In such a case, do they intersect?
3. (1 point) Find the point of intersection of \(l_1\) and \(l_2\), for \(\lambda = 1\).

\[
\text{Sol:} \quad M = \begin{bmatrix} 2\lambda & \lambda \\ 2 & -1 \end{bmatrix}
\]

\[
\det(M) = -2\lambda - 2\lambda = -4\lambda
\]

1. If \(\lambda \neq 0\), then \(\det(M) \neq 0\) and so the two lines intersect in precisely one point.

2. If \(\lambda = 0\), then \(\det(M) = 0\) and the two lines are parallel. We have

\[ l_1 : \ 0 = 1 \quad \text{\(l_1\) does not make sense!} \]

\[ \Rightarrow \text{No } \lambda. \]

3. \(\lambda = 1\) \Rightarrow \begin{cases} 2x + y = 1 \\ 2x - y = 0 \end{cases} \Rightarrow \begin{cases} 2y = 1 \\ 2x = y \end{cases} \Rightarrow \begin{cases} y = \frac{1}{2} \\ x = \frac{1}{4} \end{cases} \Rightarrow \left( \frac{1}{4}, \frac{1}{2} \right) \]