THE MODULE STRUCTURE OF HOCHSCHILD HOMOLOGY IN SOME EXAMPLES

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ABSTRACT. In this note we give a simple proof of a conjecture by A. Căldăraru stating the compatibility between the modified Hochschild–Kostant–Rosenberg isomorphism and the action of Hochschild cohomology on Hochschild homology in the case of Calabi–Yau manifolds and smooth projective curves.

1. INTRODUCTION

Let X be a smooth projective variety over the complex numbers and let $\Delta\colon X \to X \times X$ denote the diagonal embedding. The *Hochschild cohomology* ring of X is defined as

$$\mathrm{HH}^*(X) := \mathbb{H}^*(\mathcal{U}),$$

where

$$\mathcal{U} := \mathbf{R}p_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X\times X}}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathbf{L}\Delta^*\Delta_*\mathcal{O}_X, \mathcal{O}_X),$$

the morphism $p : X \times X \to X$ is one of the two projections and the multiplication $\cup$ is induced by the composition. On the other hand, the *Hochschild homology* of X is

$$\mathrm{HH}_*(X) := \mathbb{H}^{-*}(\mathcal{F}),$$

where

$$\mathcal{F} := \mathbf{L}\Delta^*\Delta_*\mathcal{O}_X.$$

The module structure over $\mathrm{HH}^*(X)$ on $\mathrm{HH}_*(X)$ is simply induced by the action $\cap\colon \mathcal{U} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ coming from the duality between $\mathcal{U}$ and $\mathcal{F}$ (see [2, 8]). More precisely, for any $i, j \in \mathbb{Z}$, the action is given by a morphism

$$\cap\colon \mathrm{HH}^i(X) \otimes \mathrm{HH}_j(X) \to \mathrm{HH}_{j-i}(X).$$

Denote by $\mathcal{T}_X$ the tangent sheaf on X. The natural morphism $I\colon S^*(\mathcal{T}_X[-1]) \xrightarrow{\sim} \mathcal{U}$ given by the adjoint of the universal Atiyah class (see [2]) is an isomorphism in $\mathrm{D^b}(X) := \mathrm{D^b}(\mathbf{Coh}(X))$ and it is known as the *Hochschild–Kostant–Rosenberg isomorphism.* On the level of cohomology, this induces an isomorphism

$$I\colon \mathrm{HT}^*(X) := \mathbb{H}^*(S^*(\mathcal{T}_X[-1])) \xrightarrow{\sim} \mathrm{HH}^*(X).$$

The wedge product yields a ring structure on $S^*(\mathcal{T}_X[-1])$ and hence on $\mathrm{HT}^*(X)$, but the map I in general is not a isomorphism of rings. It was Kontsevich's insight that the modified isomorphism

$$I^K := \mathrm{td}^{-1/2}(X) \lrcorner I^{-1}\colon \mathrm{HH}^*(X) \xrightarrow{\sim} \mathrm{HT}^*(X)$$

preserves the ring structures (see [4, 1]). The differential operator $\lrcorner$ contracts a polyvector field with a differential form and $\mathrm{td}(X)$ is the Todd class of the tangent sheaf of X.

If Ω_X is the cotangent sheaf on X, the dual of I gives an isomorphim $E\colon \mathcal{F} \xrightarrow{\sim} S^*(\Omega_X[1])$ in $\mathrm{D^b}(X)$ which in turn induces the *Hochschild–Kostant–Rosenberg isomorphism*

$$E\colon \mathrm{HH}_*(X) \xrightarrow{\sim} \mathrm{H}\Omega_*(X) := \mathbb{H}^{-*}(S^*(\Omega_X[1])).$$

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Again, one modifies E getting the isomorphism

$$I_K := \mathrm{td}^{1/2}(X) \wedge E \colon \mathrm{HH}_*(X) \xrightarrow{\sim} \mathrm{H}\Omega_*(X).$$

Locally, $I_K \colon \mathcal{F} \xrightarrow{\sim} S^*(\Omega_X[1])$ is the dual of $(I^K)^{-1} \colon S^*(\mathcal{T}_X[-1]) \xrightarrow{\sim} \mathcal{U}$ in $\mathrm{D^b}(X)$. The contraction $\lrcorner$ gives an action of $S^*(\mathcal{T}_X[-1])$ on $S^*(\Omega_X[1])$.

Conjecture 1.1. (Căldăraru) *The isomorphisms* $I^K \colon \mathrm{HH}^*(X) \xrightarrow{\sim} \mathrm{HT}^*(X)$ *and* $I_K \colon \mathrm{HH}_*(X) \xrightarrow{\sim} \mathrm{H}\Omega_*(X)$ *are compatible with the module structures on* $\mathrm{HH}_*(X)$ *and* $\mathrm{H}\Omega_*(X)$.

This is the last unsolved part of a conjecture in [2]. The other pieces of it were treated in [1, 4, 5, 6]. The previous conjecture has been proved in [3] when X has trivial canonical bundle, using a result in [6]. In turn, the proof of the latter is rather technical. The purpose of this note is to give a simple proof of the following result, relaying only on [4, 1]:

Theorem 1.2. *Conjecture 1.1 is true when X has trivial canonical bundle or has dimension* 1.

It may be worth pointing out that, also in the Calabi–Yau case, our proof differs completely from the one in [3].

2. The Proof

In the case X is a projective space $\mathbb{P}^n$, Conjecture 1.1 can be easily proved by elementary means. Indeed $\mathrm{HH}_*(\mathbb{P}^n) = \mathrm{HH}_0(\mathbb{P}^n)$. Since the action of $\mathrm{HH}^*(\mathbb{P}^n)$ on $\mathrm{HH}_*(\mathbb{P}^n)$ is graded, the only thing one has to check is that $\mathrm{HH}^0(\mathbb{P}^n) = \mathbb{C} \cdot \mathrm{id}_{\mathrm{HH}^*(\mathbb{P}^n)}$ acts compatibly with the modified Hochschild–Kostant–Rosenberg isomorphism I^K. But this is clear since, by [4, 1], I^K is a ring isomorphism and maps the identity to the identity. The same proof applies to all smooth projective varieties X whose derived category $\mathrm{D^b}(X)$ is generated by a strong exceptional collection (e.g. quadric hypersurfaces).

For the other cases, assume that X is smooth and projective. Denote by $\langle -,- \rangle_{\mathrm{HH}} \colon \mathcal{U} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{F} \to \mathcal{O}_X$ the duality pairing. Analogously, $\langle -,- \rangle_H$ is the duality pairing $S^*(\mathcal{T}_X[-1]) \otimes_{\mathcal{O}_X} S^*(\Omega_X[1]) \to \mathcal{O}_X$. By definition, the following diagram

$$\begin{array}{ccc} \mathcal{U} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\langle -,- \rangle_{\mathrm{HH}}} & \mathcal{O}_X \\ \downarrow{\scriptstyle I^K \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} I_K} & & \| \\ S^*(\mathcal{T}_X[-1]) \otimes_{\mathcal{O}_X} S^*(\Omega_X[1]) & \xrightarrow{\langle -,- \rangle_H} & \mathcal{O}_X \end{array}$$

commutes in $\mathrm{D^b}(X)$. Passing to cohomology, we get the commutative diagram

$$\begin{array}{ccc} \mathrm{HH}^*(X) \otimes \mathrm{HH}_*(X) & \xrightarrow{\langle -,- \rangle_{\mathrm{HH}}} & H^*(\mathcal{O}_X) \\ \downarrow{\scriptstyle I^K \otimes I_K} & & \| \\ \mathrm{HT}^*(X) \otimes \mathrm{H}\Omega_*(X) & \xrightarrow{\langle -,- \rangle_H} & H^*(\mathcal{O}_X), \end{array}$$

which can equivalently be rewritten as

$$\begin{array}{ccc} \mathrm{HH}_*(X) & \longrightarrow & \mathrm{HH}^*(X)^\vee \otimes H^*(\mathcal{O}_X) \\ \downarrow{\scriptstyle I_K} & & \downarrow{\scriptstyle ((I^K)^\vee)^{-1} \otimes \mathrm{id}} \\ \mathrm{H}\Omega_*(X) & \xrightarrow{\eta} & \mathrm{HT}^*(X)^\vee \otimes H^*(\mathcal{O}_X). \end{array} \tag{2.1}$$

To prove Theorem 1.2, we have to show that the diagram

$$\begin{array}{ccc} \mathrm{HH}^*(X)\otimes \mathrm{HH}_*(X) & \xrightarrow{\ \cap\ } & \mathrm{HH}_*(X) \\ \downarrow{\scriptstyle I^K\otimes I_K} & & \downarrow{\scriptstyle I_K} \\ \mathrm{HT}^*(X)\otimes \mathrm{H\Omega}_*(X) & \xrightarrow{\ \lrcorner\ } & \mathrm{H\Omega}_*(X) \end{array} \tag{2.2}$$

is commutative. For this, assume that η in (2.1) is injective. Then (2.2) is commutative if and only if the following diagram is commutative

$$\begin{array}{ccccc} \mathrm{HH}^*(X)\otimes \mathrm{HH}_*(X) & \xrightarrow{\ \cap\ } & \mathrm{HH}_*(X) & \longrightarrow & \mathrm{HH}^*(X)^\vee\otimes H^*(\mathcal{O}_X) \\ \downarrow{\scriptstyle I^K\otimes I_K} & & \downarrow{\scriptstyle I_K} & & \downarrow{\scriptstyle ((I^K)^\vee)^{-1}\otimes \mathrm{id}} \\ \mathrm{HT}^*(X)\otimes \mathrm{H\Omega}_*(X) & \xrightarrow{\ \lrcorner\ } & \mathrm{H\Omega}_*(X) & \xrightarrow{\ \eta\ } & \mathrm{HT}^*(X)^\vee\otimes H^*(\mathcal{O}_X). \end{array} \tag{2.3}$$

Notice that, more or less by definition, one has

$$\langle -,-\rangle_{\mathrm{HH}}\circ(\mathrm{id}\overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X}(-\cap -)) = \langle -,-\rangle_{\mathrm{HH}}\circ((-\cup -)\overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X}\mathrm{id})\colon \mathcal{U}\overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X}\mathcal{U}\overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X}\mathcal{F}\to\mathcal{O}_X$$

in $\mathrm{D^b}(X)$. Similarly

$$\langle -,-\rangle_H\circ(\mathrm{id}\otimes_{\mathcal{O}_X}(-\lrcorner -)) = \langle -,-\rangle_H\circ((-\wedge -)\otimes_{\mathcal{O}_X}\mathrm{id}).$$

Hence, passing to cohomologies, the main result of [1] (see also [4]) ensures that (2.3) commutes, provided that η is injective.

The Calabi–Yau case. Consider the composition with the natural projection

$$\mathrm{H\Omega}_*(X)\xrightarrow{\eta}\mathrm{HT}^*(X)^\vee\otimes H^*(\mathcal{O}_X)\twoheadrightarrow \mathrm{HT}^*(X)^\vee\otimes H^n(\mathcal{O}_X),$$

where n is the dimension of X and the canonical sheaf of X is trivial. By Serre duality, an easy computation shows that this map is an isomorphism. Hence, the map η is injective when the canonical sheaf of X is trivial.

The curve case. If X is a smooth projective curve of genus 0 or 1, the result follows from what we observed at the beginning of this section and from the previous case. Hence we can suppose that the genus of X is greater than 1. Under this assumption, we just need to check three non-trivial cases. Namely the actions $\mathrm{HH}^2(X)\otimes\mathrm{HH}_1(X)\to\mathrm{HH}_{-1}(X)$, $\mathrm{HH}^1(X)\otimes\mathrm{HH}_0(X)\to\mathrm{HH}_{-1}(X)$, and $\mathrm{HH}^1(X)\otimes\mathrm{HH}_1(X)\to\mathrm{HH}_0(X)$.

To deal with the first two cases, we just need to show that $\eta|_{\mathrm{H\Omega}_{-1}(X)}$ is injective. This is proved as follows. Pick a non-trivial $f\in\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{O}_X,\omega_X)$, where ω_X is the canonical sheaf. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{O}_X,\omega_X)\otimes\mathrm{Hom}_{\mathrm{D^b}(X)}(\omega_X,\mathcal{O}_X[1]) & \xrightarrow{\ \alpha\ } & \mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{O}_X,\mathcal{O}_X[1]) \\ \downarrow{\scriptstyle \mathrm{id}\otimes(f[1]\circ(-))} & & \downarrow{\scriptstyle f[1]\circ(-)} \\ \mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{O}_X,\omega_X)\otimes\mathrm{Hom}_{\mathrm{D^b}(X)}(\omega_X,\omega_X[1]) & \xrightarrow{\ \beta\ } & \mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{O}_X,\omega_X[1]), \end{array} \tag{2.4}$$

where the horizontal maps are given by composition. By Serre duality, β is a non degenerate pairing. Since α is nothing but (a graded piece of) the pairing $\langle -,-\rangle_H$, it is straightforward to see that the morphism η is injective.

The last case follows from the following general lemma since $\mathrm{HH}^1(X)\cong H^1(\mathcal{O}_X)$, for a curve of genus greater than 1.

Lemma 2.1. *Let X be a smooth projective variety. Then the following diagram commutes*

$$\begin{array}{ccc} H^*(\mathcal{O}_X) \otimes \mathrm{H}\Omega_*(X) & \xrightarrow{\ \lrcorner\ } & \mathrm{H}\Omega_*(X) \\ \Big\downarrow{\scriptstyle \varphi\otimes(I_K)^{-1}} & & \Big\downarrow{\scriptstyle (I_K)^{-1}} \\ \mathrm{HH}^*(X) \otimes \mathrm{HH}_*(X) & \xrightarrow{\ \cap\ } & \mathrm{HH}_*(X), \end{array}$$

where $\varphi\colon \mathcal{O}_X = \bigwedge^0 \mathcal{T}_X \hookrightarrow S^*(\mathcal{T}_X[-1]) \xrightarrow{(I^K)^{-1}} \mathcal{U}$.

Proof. First of all, observe that φ is a morphism of rings in $\mathrm{D}^{\mathrm{b}}(X)$ (see, for example, [7]). Hence the following diagram commutes in $\mathrm{D}^{\mathrm{b}}(X)$

$$\begin{array}{ccc} \mathcal{O}_X \otimes_{\mathcal{O}_X} S^*(\Omega_X[1]) & \xrightarrow{\ \lrcorner\ } & S^*(\Omega_X[1]) \\ \Big\downarrow{\scriptstyle \varphi\overset{\mathbf{L}}{\otimes}(I_K)^{-1}} & & \Big\downarrow{\scriptstyle (I_K)^{-1}} \\ \mathcal{U} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\ \cap\ } & \mathcal{F}. \end{array}$$

Taking cohomology, one gets the desired commutative diagram. □

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