Exercises
April 21, 2010

Exercise 1. Let $f(x,y) = \ln(x^2 + y^2 + 1)$.

(1) Find all **local** minima and maxima of f on its domain.
(2) Find all **global** minima and maxima of f on its domain.
(3) Find the tangent plane to the surface $z = f(x,y)$ at $(x_0, y_0) = (1,0)$ and find its normal vector.
(4) Evaluate

$$\int_S f(x,y)\, dxdy,$$

where

$$S = \left\{(x,y) \,:\, x^2 + y^2 \leq 9,\, y \geq 0\right\}.$$

(5) Evaluate

$$\int_C \nabla f \cdot d\mathbf{r},$$

where

$$C \,:\, \begin{cases} x = \log(t^2+1) \\ y = \frac{t^6}{(e-1)^3} \end{cases} \qquad 0 \leq t \leq \sqrt{e-1}.$$

Exercise 2. Let $\mathbf{F}(x,y,z) = \cos(xy)\,\mathbf{i} + \sin(xy)\,\mathbf{j} + z\,\mathbf{k}$.

(1) Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.
(2) Is $\mathbf{F}$ conservative?
(3) Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where

$$C \,:\, \begin{cases} x = t \\ y = 1 \\ z = e^t \end{cases} \qquad 0 \leq t \leq 1.$$

Exercise 3. Let

$$C:\begin{cases} x=\cos(t) \\ y=\sin(t) \\ z=\cos^2(t) \end{cases} \qquad 0\le t\le 2\pi$$

be a **closed** curve in 3D and let $\mathbf{F}(x,y,z)=-y\,\mathbf{i}+x\,\mathbf{j}-2xy\,\mathbf{k}$ be a vector field.

(1) Show that C is an integral curve of $\mathbf{F}(x,y,z)$, that is

$$x'(t)\,\mathbf{i}+y'(t)\,\mathbf{j}+z'(t)\,\mathbf{k}=\mathbf{F}(x(t),y(t),z(t)).$$

(2) Evaluate

$$\oint_C \mathbf{F}\cdot d\mathbf{r}.$$

(3) Let

$$C:\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases} \qquad a\le t\le b$$

be a **closed** curve in 3D and let $\mathbf{F}(x,y,z)$ be a vector field. Show that if

$$x'(t)\,\mathbf{i}+y'(t)\,\mathbf{j}+z'(t)\,\mathbf{k}=\mathbf{F}(x(t),y(t),z(t))$$

then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}\ge 0.$$

Exercise 4. Let $f(x,y)=-\frac{1}{x^2+y^2+1}+x^2+y^2$.

(1) Find all **local** minima and maxima of f on its domain.
(2) Find all **global** minima and maxima of f on its domain.
(3) Find the tangent plane to the surface $z=f(x,y)$ at $(x_0,y_0)=(1,1)$ and find its normal vector.
(4) Evaluate

$$\int_S f(x,y)\,dxdy,$$

where

$$S=\left\{(x,y)\,:\,x^2+y^2\le 4,\ y\le 0\right\}.$$

(5) Evaluate

$$\int_C \nabla f\cdot d\mathbf{r},$$

where

$$C:\begin{cases} x=t^{5/3} \\ y=|t| \end{cases} \qquad 0\le t\le 1.$$

Exercise 5. Let $\mathbf{F}(x, y, z) = 2yz\,\mathbf{i} + 2xz\,\mathbf{j} + 2xy\,\mathbf{k}$.

(1) Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.
(2) Show that $\mathbf{F}$ conservative.
(3) Find a scalar function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

Exercise 6. Let

$$C : \begin{cases} x = \sin(2t) \\ y = 2 \\ z = \cos(2t) \end{cases} \qquad 0 \leq t \leq \pi$$

be a **closed** curve in 3D and let $\mathbf{F}(x, y, z) = \frac{1}{4}yz\,\mathbf{i} - \frac{1}{4}yx\,\mathbf{k}$ be a vector field.

(1) Show that

$$(x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}) \cdot \mathbf{F}(x(t), y(t), z(t)) = 1.$$

(2) Evaluate

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

(3) Let

$$C : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad a \leq t \leq b$$

be a **closed** curve in 3D and let $\mathbf{F}(x, y, z)$ be a vector field. Show that if

$$(x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}) \cdot \mathbf{F}(x(t), y(t), z(t)) = 1$$

then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = b - a.$$

Use this to show that $\mathbf{F}$ is not conservative.

Exercise 7. Let $f(x, y) = e^{x^3+y^3-3x-3y}$.

(1) Find all **local** minima and maxima of f on its domain.
(2) Show that f has no **global** maximum. (**Suggestion**: Consider the restriction of $f(x, y)$ to the line $(x, y) = (t, t)$ and let t go to $+\infty$.)
(3) Find the tangent plane to the surface $z = f(x, y)$ at $(x_0, y_0) = (0, 0)$.

(4) Evaluate

$$\int_R (x^2-1)(y^2-1)f(x,y)\,dxdy,$$

where R is the rectangle

$$R = \{(x,y)\,:\,0 \le x \le 1,\ 0 \le y \le 1\}\,.$$

(5) Evaluate

$$\int_C \nabla f \cdot d\mathbf{r},$$

where

$$C\,:\begin{cases} x = t^{100} \\ y = \log(t^2 - t + 1) \end{cases} \qquad 0 \le t \le 1.$$

Exercise 8. Let $\mathbf{F}(x,y) = (6x^5 + ye^{xy})\,\mathbf{i} + (5y^4 + xe^{xy})\,\mathbf{j}$.

(1) Compute $\nabla \cdot \mathbf{F}$.
(2) Show that $\mathbf{F}$ conservative.
(3) Find a scalar function $f(x,y)$ such that $\nabla f = \mathbf{F}$.

Exercise 9. Let

$$C\,:\begin{cases} x = \frac{1}{\sqrt{2}}\sin(2t) + 1 \\ y = \frac{1}{\sqrt{2}}\sin(2t) + 2 \\ z = \cos(2t) + 1 \end{cases} \qquad 0 \le t \le \pi$$

be a **closed** curve in 3D and let $\mathbf{F}(x,y,z) = \frac{2}{\sqrt{2}}z\,\mathbf{i} + \frac{2}{\sqrt{2}}z\,\mathbf{j} - \sqrt{2}(x+y)\,\mathbf{k}$ be a vector field.

(1) Show that

$$\mathbf{F}(x(t),y(t),z(t)) = (x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}) + \left(\frac{2}{\sqrt{2}}\,\mathbf{i} + \frac{2}{\sqrt{2}}\,\mathbf{j} + -3\sqrt{2}\,\mathbf{k}\right).$$

(2) Evaluate

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

(3) Let

$$C\,:\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad a \le t \le b$$

be a **closed** curve in 3D and let $\mathbf{F}(x, y, z)$ be a vector field. Show that if

$$\mathbf{F}(x(t), y(t), z(t)) = (x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}) + \mathbf{v},$$

where $\mathbf{v}$ is a costant vector, and

$$|x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}| = 1,$$

then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = b - a.$$

(**Suggestion**: Use that the constant vector field $\mathbf{v}$ is conservative.)