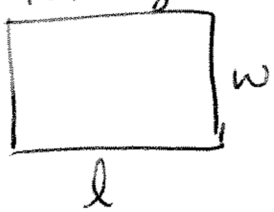
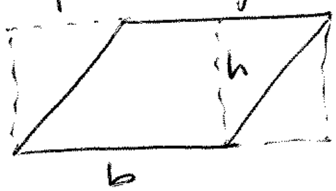
4.1 Introduction to Area

rectangle



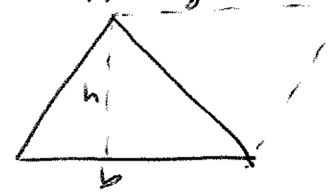
$A = lw$

parallelogram

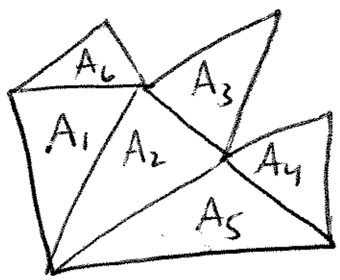


$A = bh$

triangle

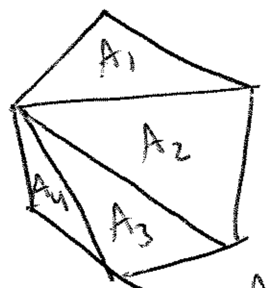


$A = \frac{1}{2}bh$

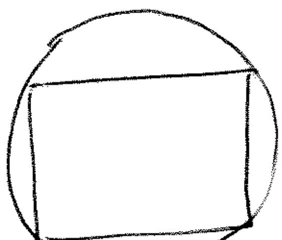


polygon

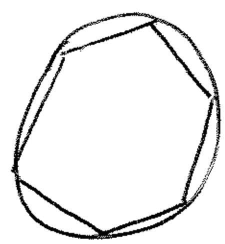
$A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$



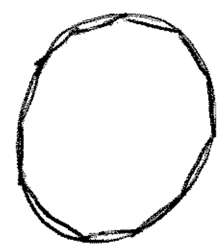
$A = A_1 + A_2 + A_3 + A_4$



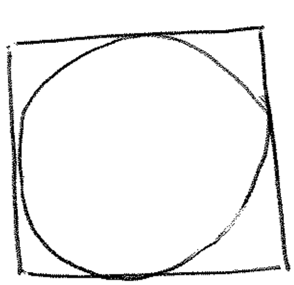
Estimate area of circle w/ inscribed rectangle.



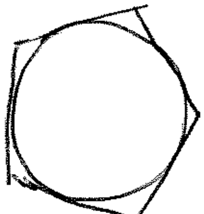
Estimate area better w/ hexagon



Estimate area of circle w/ inscribed polygon of increasing # of sides.



Estimate area of circle with circumscribed square.



Estimate area w/ pentagon.

Estimate area of circle w/ circumscribed n-gon.

4.1 (Sums + Sigma Notation)

$1+2+3+4+\dots+100 = \sum_{i=1}^{100} i$

OR $2+4+6+8+\dots+1000 = \sum_{i=1}^{500} 2i$

OR $1+4+9+16+\dots+625 = \sum_{i=1}^{25} i^2$

sequence

$\{a_i\}$ notation

$a_1, a_2, a_3, \dots$

e.g.

$1, 1, 2, 3, 5, 8, \dots$

$\sum$ = capital Greek symbol called "sigma"; it means summation

i = index (we can also call this j or k, or whatever)

$$\sum_{j=1}^{n} \frac{1}{j} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$$\boxed{\sum_{i=1}^{n} c = \underbrace{c+c+c+\dots+c}_{n \text{ times}} = nc}$$

where c is independent of index

Linearity of $\sum$

Let $\{a_i\}$ + $\{b_i\}$ denote 2 sequences + $c \in \mathbb{R}$. Then,

(i) $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

+ (ii) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

4.1 (continued)

Special Sum Formulas *(see proofs pg 225) for ① + ②

① $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

② $\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

③ $\sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

④ $\sum_{i=1}^{n} i^4 = 1^4+2^4+3^4+\cdots+n^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$

Convincing argument from Gauss.

4.1 (Continued)

$$\boxed{\text{Collapsing Sum} \quad \sum_{i=1}^{n} (a_{i+1} - a_i) = a_{n+1} - a_1}$$

Ex 1 $\quad \displaystyle\sum_{k=1}^{10} (2^k - 2^{k-1})$

Ex 2 $\quad \displaystyle\sum_{k=3}^{m+1} (a_k - a_{k-1})$

4.1 (continued)

Ex 3 $\sum_{i=1}^{10} [(i-1)(4i+3)]$

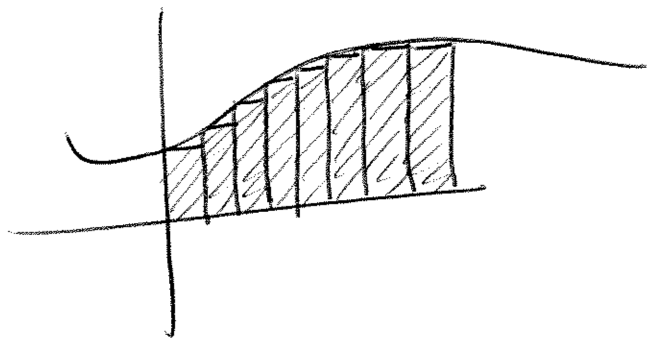
Ex 4 $\sum_{i=1}^{7} (2i-3)^2$

Ex 5 (#32) change the variable in the index to start at 1.

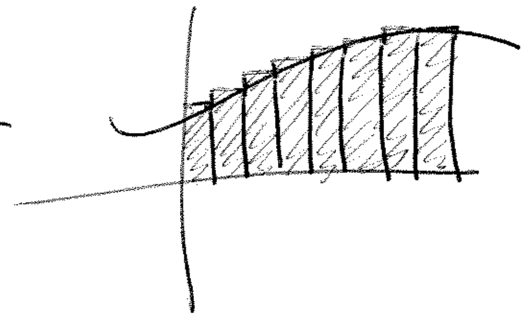
$$\sum_{k=5}^{14} k2^{k-4}$$

4.1 (continued)

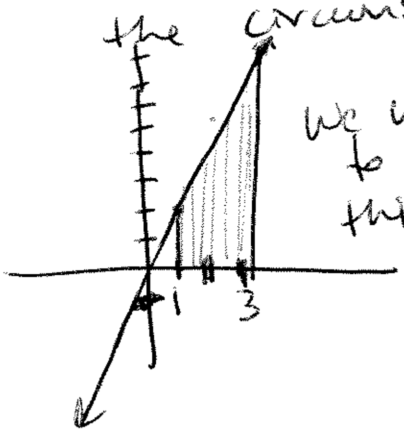
To find area under curve, we can estimate it w/ inscribed or circumscribed rectangles.



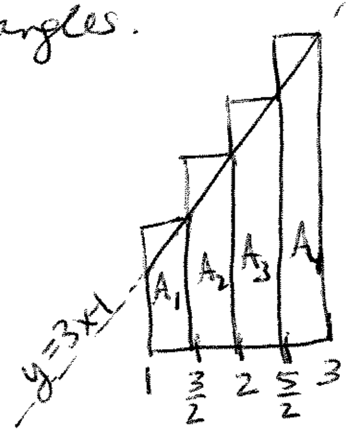
OR



Ex 6 For $f(x) = 3x-1$, divide the interval $[1, 3]$ into 4 equal subintervals. Calculate area of the circumscribed rectangles.



We want to estimate this area.



Here's our estimation.

$$A = A_1 + A_2 + A_3 + A_4$$

$$= \left(\tfrac{1}{2}\right)\left(f\left(\tfrac{3}{2}\right)\right) + \left(\tfrac{1}{2}\right)\left(f(2)\right) + \tfrac{1}{2}\left(f\left(\tfrac{5}{2}\right)\right) + \tfrac{1}{2}\left(f(3)\right)$$

base * ht + base * ht + base * ht + base * ht

$$= \tfrac{1}{2}\left(3\left(\tfrac{3}{2}\right)-1\right) + \tfrac{1}{2}\left(3(2)-1\right) + \tfrac{1}{2}\left(3\left(\tfrac{5}{2}\right)-1\right) + \tfrac{1}{2}\left(3(3)-1\right)$$

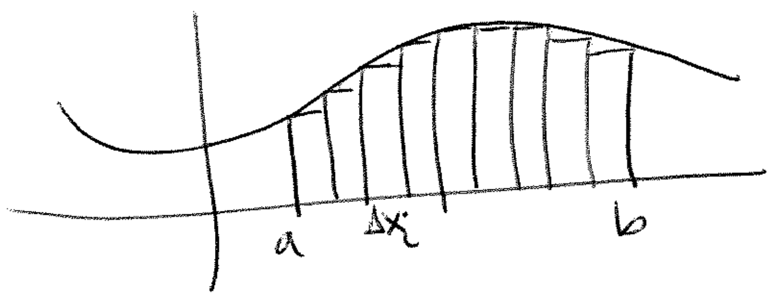
$$= \tfrac{1}{2}\left(\tfrac{7}{2}\right) + \tfrac{1}{2}(5) + \tfrac{1}{2}\left(\tfrac{13}{2}\right) + \tfrac{1}{2}(8)$$

$$= \tfrac{7}{4} + \tfrac{10}{4} + \tfrac{13}{4} + \tfrac{16}{4}$$

$$= \tfrac{46}{4} = \tfrac{23}{2}$$

4.2 The Definite Integral

In general, we can find the area under a curve using inscribed or circumscribed rectangles.



The area of each rectangle is base × ht, where base = Δx_i + height = $f(x_i)$ (i = index).

$$A = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_n)\Delta x_n$$

If we divide our $[a,b]$ interval into n equal subintervals, then $\Delta x_1 = \Delta x_2 = \ldots = \Delta x_n = \frac{b-a}{n}$ and

$x_1 = a + \frac{b-a}{n}$ $\quad x_2 = x_1 + \frac{b-a}{n} = a + 2\left(\frac{b-a}{n}\right)$

$\ldots$ $x_i = a + i\left(\frac{b-a}{n}\right)$ $\ldots$ $x_n = a + n\left(\frac{b-a}{n}\right) = b$

Riemann sum

$$\boxed{A \cong \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) \text{ where } x_i = a + i\left(\frac{b-a}{n}\right)}$$

If we let n continue to grow, our estimation will be better, until it becomes exact, i.e.

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}$$

☆ Area is a signed area, i.e. area above x-axis is +ve + area below x-axis is −ve.

Defn Definite Integral

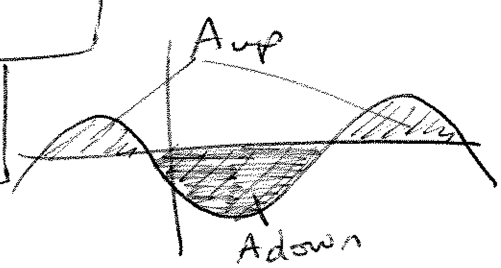
Let f be a functn that's defined on $[a,b]$. If

$\lim_{|P| \to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i$ exists, we say f is integrable

on $[a,b]$.

$$\underbrace{\int_a^b f(x)\,dx}_{\text{called definite integral}} = \lim_{|P| \to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i$$

$$\int_a^b f(x)\,dx = A_{up} - A_{down}$$



$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad a > b$$

Integrability Thm

If f is bounded on $[a,b]$ & continuous there except for a finite # of discontinuities, then f is integrable on $[a,b]$. So, if f is continuous on $[a,b]$, it's integrable on $[a,b]$.

Interval Additive Property If $f(x)$ integrable,

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

4.2 (continued)

Ex 1 Use $f(x) = 3x-1$, on the interval $[1,3]$.
Subdivide it into 4 equal subintervals & Calculate the area of the inscribed rectangles.

Ex 2 Evaluate the definite integral using the defn.

$$\int_{-1}^{2} (x^2+1)\,dx$$

$$a=-1,\ b=2 \Rightarrow \Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$x_i = -1 + i\left(\frac{3}{n}\right) = -1 + \frac{3}{n}i$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2+1)\left(\frac{3}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left[-1+\frac{3}{n}i\right]^2+1\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1-\frac{3}{n}i-\frac{3}{n}i+\frac{9}{n^2}i^2+1\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2-\frac{6}{n}i+\frac{9}{n^2}i^2\right)\frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{6}{n}-\frac{18}{n^2}i+\frac{27}{n^3}i^2\right)$$

$$= \lim_{n\to\infty}\left[\frac{6}{n}\sum_{i=1}^{n}1 - \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{6}{n}(n) - \frac{18}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty}\left[6 - \frac{9(n+1)}{n} + \frac{9}{2n^2}(n+1)(2n+1)\right]$$

$$= \lim_{n\to\infty}\left[6 - 9 - \frac{9}{n} + \frac{9}{2n^2}(2n^2+3n+1)\right]$$

$$= \lim_{n\to\infty}\left[-3 - \frac{9}{n} + 9 + \frac{27}{2n} + \frac{9}{2n^2}\right]$$

$$= -3+9 = \boxed{6}$$

4.2 (continued)

Ex 3 Evaluate the definite integral using the definition. $\int_{-2}^{1} (3x^2+2)\,dx$

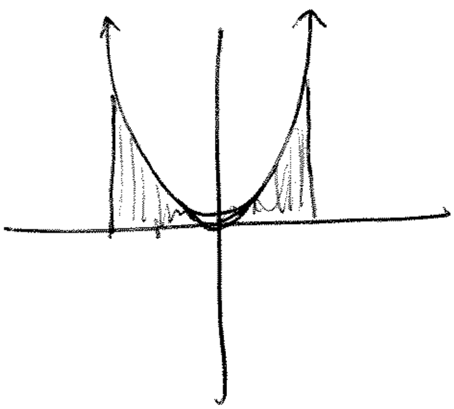
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$\Delta x =$

$x_i =$

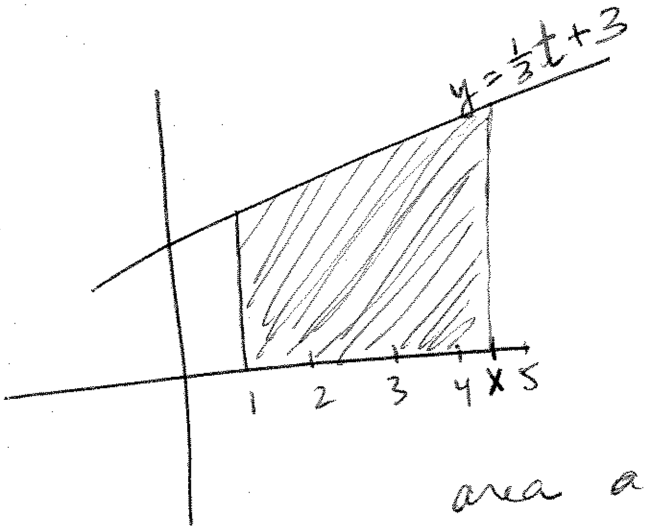
$$= \lim_{n\to\infty} \sum_{i=1}^{n} (3x_i^2+2)\,\Delta x$$

Ex 4 Find the area of the region under the curve of $f(x) = x^2$ on the interval $[-2,2]$. (To do this, divide the interval $[-2,2]$ into n equal subintervals, calulate the area of the circumscribed or inscribed rectangles + take limit as $n \to \infty$.)



4.3 The First Fundamental Theorem of Calculus

Accumulation Functions



If we want to find area of shaded region, it's

$$A=\int_1^x\left(\tfrac{1}{3}t+3\right)dt.$$

This is an accumulation function, because it accumulates area as we move x.

What is derivative of A?

Well, A is area of a trapezoid.

$y(1)=\frac{1}{3}+3=\frac{10}{3}$

$y(x)=\frac{1}{3}x+3$

$$\Rightarrow \quad A=\frac{1}{2}\underbrace{\left(\frac{10}{3}+\frac{1}{3}x+3\right)}_{\text{sum of bases}}\underbrace{(x-1)}_{\text{height of trapezoid}}$$

$$A=\frac{1}{2}\left(\frac{10}{3}x-\frac{10}{3}+\frac{1}{3}x^2-\frac{1}{3}x+3x-3\right)$$

$$A=\frac{1}{2}\left(\frac{1}{3}x^2+6x-\frac{19}{3}\right)$$

$$A(x)=\frac{1}{6}x^2+3x-\frac{19}{6}$$

$$\Rightarrow \quad A'(x)=\frac{1}{3}x+3$$

Aha! $\quad A'(x)=\frac{d}{dx}\left[\int_1^x\left(\frac{1}{3}t+3\right)dt\right]=\frac{1}{3}x+3$ ☺

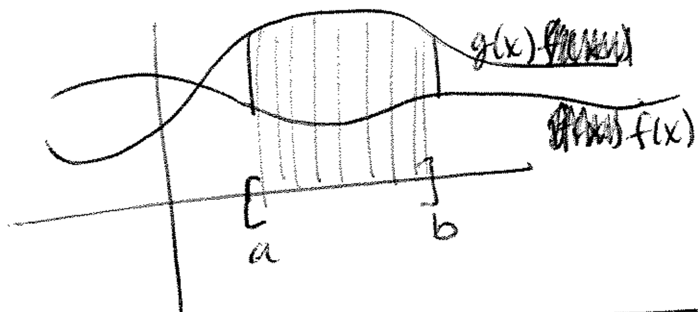
Thm First Fundamental Thm of Calculus

Let f be continuous on $[a,b]$ + let x be a value in (a,b). Then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Thm B Comparison Property

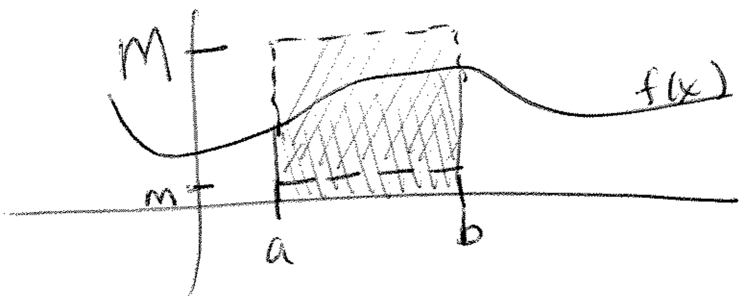
If f + g are integrable on $[a,b]$ + if $f(x) \le g(x)$ $\forall x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.



i.e. the area under $f(x)$ on $[a,b]$ is less than (or =) the area under $g(x)$ on $[a,b]$.

Thm Boundedness Property

If f is integrable on $[a,b]$ + $m \le f(x) \le M$ $\forall x \in [a,b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$



i.e. the area under $f(x)$ is smaller than the big rectangle but bigger than the small rectangle.

4.3 (continued)

Thm Linearity of Definite Integral

If f & g are integrable on $[a,b]$ & $k \in \mathbb{R}$,

(i) $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$

and (ii) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Ex 1 (#12) Suppose $\int_0^1 f(x)\,dx = 2$ $\int_1^2 f(x)\,dx = 3$

$\int_0^1 g(x)\,dx = -1$ and $\int_0^2 g(x)\,dx = 4$.

Calculate $\int_0^2 [\sqrt{3} f(t) + \sqrt{2} g(t) + \pi]\,dt$.

4.3 (continued)

Ex 2 Find $G'(x)$.

(a) $G(x) = \int_3^x 4t\,dt$

(b) $G(x) = \int_1^x \cos^3(2t)\tan(t)\,dt$ $\quad -\pi/2 < x < \pi/2$

(c) $G(x) = \int_1^x xt\,dt$ (Tricky)

4.3 (continued)

Ex 3 Find $\frac{d}{dx}\int_1^{x^2+x} \sqrt{2w+\sin w}\, dw$

4.4 The Second Fundamental Thm of Calculus + The Method of Substitution

Second Fundamental Thm of Calculus

Let f be continuous on $[a,b]$ + F be any antiderivative of f on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Ex 1 $\int_{-1}^{2} x^4 \, dx$

Ex 2 $\int_{\pi/6}^{\pi/2} 2 \sin t \, dt$

4.4 (continued)

Substitution Rule for Indefinite Integrals

Let g be differentiable and F = antiderivative of f. Then, if $u = g(x)$,

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u) + C$$
$$= F(g(x)) + C$$

Our friendly u-substitution is now formalized for more than the "power rule" functions.

Ex 3 $\displaystyle\int \sqrt{x^3+1}\,(3x^2)\,dx$

Ex 4 $\displaystyle\int_0^{\pi/2} \sin^2(3x)\cos(3x)\,dx$

4.4 (continued)

Ex 5 $\displaystyle\int_1^3 \frac{x^2+1}{\sqrt{x^3+3x}}\,dx$

Ex 6 $\displaystyle\int_{-4}^{-1} \frac{1-s^4}{2s^2}\,ds$

4.5 Mean Value Theorem (MVT) for Integrals & Symmetry

Defn Avg Value of a Function

If f is integrable on $[a,b]$, then the average value of f on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex 1 Find the average value of

$$f(x)=\frac{x}{\sqrt{x^2+16}} \quad \text{on } [0,3].$$

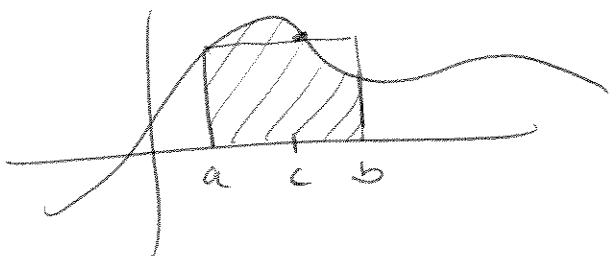
Mean Value Thm for Integrals

If f is continuous on $[a,b]$, $\exists$ a # $c \in (a,b) \ni$

$$\int_a^b f(t)\,dt = f(c)(b-a)$$

i.e. $\exists$ some $c \in (a,b) \ni$ rectangle with height $f(c)$ + width $(b-a)$ is $=$ area under curve on $[a,b]$

$$\Rightarrow f(c) = \frac{\int_a^b f(t)\,dt}{b-a}$$



i.e. $f(c)$ is avg (mean) value of f on $[a,b]$.

Pf let $G(x) = \int_a^x f(t)\,dt \qquad x \in [a,b]$

By MVT for derivatives (since $f(x)$ is cont, then $G(x)$ is cont. + differentiable), $\exists$ $c \in (a,b) \ni$

$$\frac{G(b)-G(a)}{b-a} = G'(c)$$

$$\Leftrightarrow \quad G(b) - G(a) = G'(c)(b-a)$$

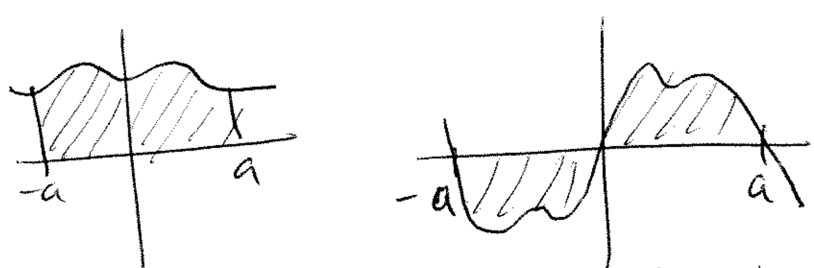
$$\int_a^b f(t)\,dt - \int_a^a f(t)\,dt = G'(c)(b-a)$$

$$\int_a^b f(t)\,dt = f(c)(b-a) \;//$$

4.5 (continued)

Ex2 Find the values of c that satisfy the MVT for $f(x) = x(1-x)$ on $[0,1]$.

Symmetry Thm
If f is even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
If f is odd function, then $\int_{-a}^{a} f(x)\,dx = 0$.

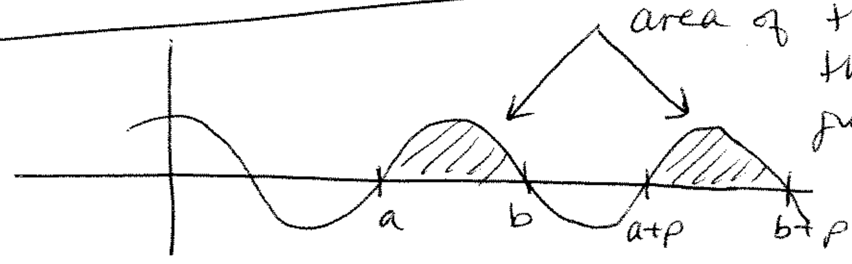


even fn   odd function

Thm P If f is periodic w/ period p, then

$$\int_{a+p}^{b+p} f(x)\,dx = \int_{a}^{b} f(x)\,dx$$

area of these are the same, they're just located differently



4.5 (continued)

Ex 3 $\int \sin(2x-4)\,dx$

Ex 4 $\int x^4 \cos(\pi x^5 - \sqrt{7})\,dx$

Ex 5 $\int x^6 (7x^7+\pi)^8 \sin\left[(7x^7+\pi)^9\right]dx$

4.5 (continued)

Ex 6 $\displaystyle\int_0^2 \frac{x^2}{(9-x^3)^{3/2}}\,dx$

Ex 7 $\displaystyle\int_0^{\pi/2} \sin x \sin(\cos x)\,dx$

4.5 (continued)

Ex 8 $\int_{1}^{2} \left(1+\frac{1}{t}\right)^2 \left(\frac{1}{t^2}\right) dt$

Ex 9 $\int_{-\pi/2}^{\pi/2} x^2 \sin^2(x^3) \cos(x^3)\, dx$

Ex 10 $\int_{-\pi/2}^{\pi/2} x \sin^2(x^3) \cos(x^3)\, dx$