13.1 Double Integrals over Rectangles

Remember \( \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx \) is one definition of the definite integral, i.e., the area under the curve \( y = f(x) \) from \( x=a \) to \( x=b \).

Basically, we added up lots of rectangles to get our area. It should be too surprising, then, for \( z = f(x, y) \) that finding the volume under the surface requires adding up volumes of rectangular boxes. It requires choosing a rectangular region in the \( xy \)-plane and cutting it into small rectangles.

\[ \text{Volume of box} \Rightarrow V = f(x_k, y_k) \Delta A_k \]

where \( f(x_k, y_k) \) = height \( \Delta A_k \) = area of base of box = \( \Delta x \Delta y \Delta z \)

\[ \text{Volume under } z = f(x, y) \text{ over } R = \text{sum of all the rectangular boxes (like one in figure)} \]
13.1 (continued)

**Defn Double Integral**

Let \( f(x,y) \) be defined on a closed rectangle \( R \).
If \( \lim_{P \to 0} \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k) \Delta A_k \) exists, then \( f \) is integrable over \( R \), and the double integral
\[
\iint_R f(x,y) \, dA = \lim_{P \to 0} \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k) \Delta A_k.
\]

**Integrability Theorem**

If \( f \) is bounded on the closed rectangle \( R \) and if it is continuous there, except for a finite number of smooth curves, then \( f \) is integrable on \( R \). If \( f \) is continuous on all of \( R \), then \( f \) is integrable there.

**Properties of the Double Integral**

A) It's linear =
1. \( \iint_R k f(x,y) \, dA = k \iint_R f(x,y) \, dA \)
2. \( \iint_R \left[ f(x,y) + g(x,y) \right] \, dA = \iint f(x,y) \, dA + \iint g(x,y) \, dA \)

B) Additive on rectangles
\[
\iint_R f(x,y) \, dA = \iint_{R_1} f(x,y) \, dA + \iint_{R_2} f(x,y) \, dA
\]

C) If \( f(x,y) \leq g(x,y) \), then
\[
\iint_R f(x,y) \, dA \leq \iint_R g(x,y) \, dA
\]
13.1 (continued)

\[ \iiint_R k \, dA = k \, \iint_R dA = k \, A(R) \]

**Example 1**

For \( f(x,y) = \begin{cases} -1 & 1 \leq x \leq 4, \ 0 \leq y < 1 \\ 2 & 1 \leq x \leq 4, \ 1 \leq y \leq 2 \end{cases} \)

find \( \iiint_R f(x,y) \, dA \) where \( R = \{ (x,y) \mid 1 \leq x \leq 4, \ 0 \leq y \leq 2 \} \).
Ex 2. Let \( R = \{(x,y) \mid 0 \leq x \leq 6, 0 \leq y \leq 4\} \) and \( f(x,y) = 10y^2 \).

Partition \( R \) into 6 equal squares by lines \( x = 2 \), \( x = 4 \), and \( y = 2 \). Approximate \( \iint_R f(x,y) \, dA \) as \( \sum_{k=1}^6 f(x_k,y_k) \, \Delta A_k \)

where \((x_k, y_k)\) are centers of squares.
Ex 3 Calculate \( \iint_R f(x,y) \, dA \) where \( f(x,y) = x - y \).

\[ R = \{(x,y) \mid 0 \leq x \leq 2, \ 0 \leq y \leq 1\} \]

(Hint: Sketch it and see if you recognize it.)

It's a rectangular box with some of the top chopped off.

So, we have

\[ \text{Box: } 6 \quad \text{and} \quad \text{Cone: } \]

\[ 2 \]

\[ 1 \]

\[ 1 \]

\[ \frac{1}{2} \]
13.2 Iterated Integrals

We can think about the volume slightly differently.

To find this volume, we can take thin “slab” cross-sections and add them up.

Each slab has volume $A(y) \, dy$.

$\Rightarrow V = \int_{c}^{d} A(y) \, dy$ but

$A(y) = \int_{a}^{b} f(x, y) \, dx$

$\Rightarrow V = \int_{c}^{d} \left( \int_{a}^{b} f(x, y) \, dx \right) \, dy$

(If we had taken slabs parallel to the $yz$-plane, then we'd get $V = \int_{a}^{b} \left( \int_{c}^{d} f(x, y) \, dy \right) \, dx$.)

$\Rightarrow \int_{R} f(x, y) \, dA = \int_{a}^{b} \left[ \int_{c}^{d} f(x, y) \, dy \right] \, dx = \int_{c}^{d} \left[ \int_{a}^{b} f(x, y) \, dx \right] \, dy$

Ex1 Evaluate $\int_{0}^{4} \left[ \int_{-1}^{3} (x^2 - 7y) \, dx \right] \, dy$
13.2 (continued)

Ex 2 \[ \iint_{S_1} \frac{y}{(xy+1)^2} \, dx \, dy \]

Ex 3 \[ \iint_{R} xy \sqrt{1+x^2} \, dA \quad \text{where} \quad R=\{(x,y) \mid 0 \leq x \leq \sqrt{3}, 1 \leq y \leq 2\} \]
Ex 4 Find the volume of the solid in the 1st octant enclosed by \( z = 4 - x^2 - y^2 \).
13.3 Double Integrals over Non-rectangular Regions

What if the region we're integrating over is not a rectangle, but a simple, closed curve region instead?

\[ V = \iint_S f(x,y) \, dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y) \, dy \, dx \]

Ultimately, we'll have variables in our integration limits.

**Ex 1** Find \( \iint_S (x+y) \, dA \) where \( S \) is the triangle with vertices \((0,0), (0,4), (1,4)\).

\[ V = \int_0^1 \int_0^{4x} (x+y) \, dy \, dx \]

Then, we need to find limits for \( y \) first, which will be dependent on \( x \).

The line from \((0,0)\) to \((1,4)\) is \( y = 4x \), \( y \) goes from \( y = 4x \) up to \( y = 4 \). And given that, \( x \) goes from 0 to 1.

\( \Rightarrow \)

\[ V = \int_0^1 \left[ \int_0^{4x} (x+y) \, dy \right] \, dx = \int_0^1 (xy + \frac{1}{2} y^2) \bigg|_0^4 \, dx \]

\[ = \int_0^1 (4x + 8) - (0 + 0) \, dx = \int_0^1 4x + 8 \, dx \]

\[ = \left[ 2x^2 + 8x \right]_0^1 = 6 \]

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13.3 (continued)

Ex 2 Evaluate $\iiint_S x \, dA$ where $S$ is region between $y = x$ and $y = x^3$.

$0 \leq x \leq 1$

$\frac{3}{2} \leq y \leq x$

$\iiint_S x \, dA =$
Ex 3 Write these integrals as iterated integrals with the order of integration switched.

(a) $\int_0^2 \int_{y^2}^{2y} f(x,y) \, dx \, dy$

(b) $\int_{1/2}^1 \int_{x^3}^x f(x,y) \, dy \, dx$

(c) $\int_0^1 \int_{y}^y f(x,y) \, dx \, dy$
Ex 4 Evaluate

(a) $\int_1^5 \int_0^x \frac{3}{x^2+y^2} \, dy \, dx$

(b) $\int_{\pi/2}^{\pi} \int_0^{\sin \theta} r \cos \theta \, dr \, d\theta$
13.3 (continued)

Ex 5 Find the volume of the solid bounded by the parabolic cylinder $x^2 = 4y + 4$ and the planes $z = 0$ and $5y + 9z - 45 = 0$.
13.4 Double Integrals In Polar Coordinates

Rather than finding the volume over a rectangle (for Cartesian coords), we'll use a "polar rectangle" for polar coords.

The area of a sector of a circle is given by

\[ A_{\text{sector}} = \pi r^2 \left( \frac{\Delta \theta}{2\pi} \right) = \frac{1}{2} \Delta \theta r^2 \]

where \( \Delta \theta \) is the angle of the piece.

\[ = \frac{1}{2} \Delta \theta r_0^2 - \frac{1}{2} \Delta \theta r_i^2 \]

where \( r_o \) = outer rad., \( r_i \) = inner rad.

\[ = \frac{\Delta \theta}{2} (r_o^2 - r_i^2) = \frac{\Delta \theta}{2} (r_o - r_i)(r_o + r_i) \]

\[ = \Delta \theta \left( r_o - r_i \right) \left( \frac{r_o + r_i}{2} \right) \]

\[ A_{\text{polar rect}} = \Delta \theta \Delta r \bar{r} \]

where \( \bar{r} \) = avg radius

\[ \bar{r} = \frac{r_o + r_i}{2} \]

and \( \Delta r = r_o - r_i \)

\[ \Rightarrow \text{Volume of surface } f(x,y) \text{ over } R \text{ is} \]

\[ V \approx \frac{1}{2} \sum_{k=1}^{n} \left( \bar{r}_k \bar{\theta}_k \right) \bar{r}_k \Delta r_k \Delta \theta_k \]

\[ \text{Area of each polar rectangle} \]

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Ex 1 Find the area of the given region $S$ by calculating $\iint_S r\,dr\,d\theta$.

(a) $S$ is smaller region bounded by $\theta = \frac{\pi}{6}$ and $r = 4\sin\theta$.

\[
0 \leq r \leq 4\sin\theta \quad \forall \theta \in \left[0, \frac{\pi}{6}\right]
\]

\[
\iint_S r\,dr\,d\theta = \int_{\theta=0}^{\pi/6} \int_{r=0}^{4\sin\theta} r\,dr\,d\theta
\]

\[
= \int_{\theta=0}^{\pi/6} \frac{1}{2} r^2 \Big|_0^{4\sin\theta} d\theta
\]

\[
= \int_{\theta=0}^{\pi/6} 8\sin^2\theta \,d\theta
\]

\[
= \frac{8}{2} \int_{\theta=0}^{\pi/6} 1 - \cos(2\theta) \,d\theta
\]

\[
= 4 \left( \theta - \frac{1}{2} \sin(2\theta) \right) \Big|_0^{\pi/6}
\]

\[
= 4 \left( \frac{\pi}{6} - \frac{1}{2} \sin\left(\frac{\pi}{3}\right) \right) - 0
\]

\[
= 4 \left( \frac{\pi}{6} - \frac{\sqrt{3}}{2} \right) = \frac{2\pi}{3} - \sqrt{3}
\]
Ex. 1 (Cont.)

(b) \( S \) is region outside circle \( r = 2 + \) inside lemniscate \( r^2 = 9 \cos 2\theta \).

It's symmetric, so we can just double one piece.

(Refer to 10.6 in book if you need to know how to graph some of these polar graphs.)
Ex 2 Evaluate using polar cords.

(a) \( \int_0^1 \int_0^{\sqrt{1-y^2}} f(y) \, dx \, dy \) where \( S \) is 1st Quadrant polar rectangle inside \( x^2+y^2 = 1 \) and outside \( x^2+y^2 = 1 \).

(b) \( \int_0^1 \int_0^{\sqrt{1-y^2}} \sin(x^2+y^2) \, dx \, dy \)
13.4 Surface Area

To find the surface area, we're basically going to add up lots of little parallelograms that are tangent to the surface.

\[ \vec{u}_m = \Delta x_m \hat{i} + f_x(x_m, y_m) \Delta x_m \hat{k} \]
\[ \vec{v}_m = \Delta y_m \hat{j} + f_y(x_m, y_m) \Delta y_m \hat{k} \]

We know \( A(T_m) \) (the area of the parallelogram) is the magnitude of the cross product of its vector sides.

\[ \implies A(T_m) = |\vec{u}_m \times \vec{v}_m| \]

and \( \vec{u}_m \times \vec{v}_m = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x_m & 0 & f_x \Delta x_m \\ 0 & \Delta y_m & f_y \Delta y_m \end{vmatrix} \)
\[ \vec{u}_m \times \vec{v}_m = (-\Delta x_m \Delta y_m f_x(x_m, y_m)) \hat{\mathbf{i}} \]
\[ -\left( \Delta x_m \Delta y_m f_y(x_m, y_m) \right) \hat{\mathbf{j}} + \Delta x_m \Delta y_m \hat{\mathbf{k}} \]
\[ = \Delta x_m \Delta y_m \left( -f_x(x_m, y_m) \hat{\mathbf{i}} - f_y(x_m, y_m) \hat{\mathbf{j}} + \hat{\mathbf{k}} \right) \]

\[ A(T_m) = |\vec{u}_m \times \vec{v}_m| \]
\[ = \Delta x_m \Delta y_m \sqrt{[f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2} + 1 \]
\[ A(\mathcal{R}_n) \]
\[ = A(\mathcal{R}_n) \sqrt{[f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2} + 1 \]

If we add all these little tangent parallelograms together, we'll have our surface area.

\[ \Rightarrow SA = \lim_{|\mathcal{P}| \to 0} \sum_{m=1}^{N} A(T_m) = \lim_{|\mathcal{P}| \to 0} \sum_{m=1}^{N} \sqrt{[f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2} + 1 \cdot A(\mathcal{R}_n) \]

\[ SA = \iint_{S} \sqrt{f_x^2 + f_y^2 + 1} \, dA \]
Ex 1 Find the surface area of the plane that is bounded by $x=0$, $y=0$.

$3x-2y+6z=12$

$+ 3x+2y=12$ planes.
13.6 (continued)

Ex. 2 Find the surface area for part of the sphere \( x^2 + y^2 + z^2 = 9 \) inside the circular cylinder \( x^2 + y^2 = 4 \).
13.7 Triple Integrals

\[ A = \int_a^b f(x) \, dx \]  
(measures 2d space under curve \( f(x) \))

\[ V = \iiint_S f(x, y) \, dA \]  
(measures 3d space under surface \( f(x, y) \))

\[ = \) We predict that \( \iiint_S f(x, y, z) \, dV \) measures 4d space under "hyper surface" \( f(x, y, z) \).

Basically, we will extend the pattern we established for definite integrals. In 4d, we add little boxes of small volume \( \times \) "height" \( \times \) the function to get the 4d space under the hyper surface.

\[ \iiint_S f(x, y, z) \, dV = \int_a^b \int_{q_2(x)}^{q_1(x)} \int_{q_3(x, y)}^{q_2(x, y)} f(x, y, z) \, dz \, dy \, dx \]

where our integration limits are determined by our 3d region. Notice that innermost integral can depend on 2 variables, middle integral can only depend on 1 variable, and last integral can only have constants for its bounds.

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Ex 1 Write an iterated integral for \( \iiint f(x, y, z) \, dv \) where \( S \) is region in 1st octant bounded by the surface \( z = 9 - x^2 - y^2 \) and the coordinate planes.

Ex 2 Evaluate \( \iiint_{S}^{\pi/2} \int_{0}^{y} \int_{0}^{\pi/2} \sin(x + y + z) \, dx \, dy \, dz \)
13.7 (Continued)

Ex. 3. Find the volume of the solid in the first octant bounded by the elliptic cylinder $y^2 + 6z^2 = 4$ and the plane $y = x$.

1. Use method from 16.3.

2. Use $V = \iiint dx\, dy\, dz$.
Ex 4 Find the volume of the solid bounded by

\[ y = x^2 + 2, \quad y = 4, \quad z = 0 + 3y - 4z = 0 \]

\(\text{(cylinder)}\)
\(\text{(planes)}\)
13.8 Triple Integrals (Cylindrical + Spherical Coordinates)

\[ \iiint_S f(x,y,z) \, dV = \int_0^2 \int_{r_1(r)}^{r_2(r)} \int_{\theta_1}^{\theta_2} f(\rho \cos \theta, \rho \sin \theta, z) \, r \, dz \, dr \, d\theta \]

cylindrical coordinates

Ex 1 Find the volume of the solid bounded above by the sphere \( x^2 + y^2 + z^2 = 9 \), below by the plane \( z = 0 \) and laterally by the cylinder \( x^2 + y^2 = 4 \). (Use cylindrical coordinates.)
\[ \iiint f(x, y, z) \, dV = \int_0^\pi \int_0^\pi \int_0^{\sqrt{16}} f(\rho \sin \theta \cos \varphi, \rho \sin \theta \sin \varphi, \rho \cos \varphi) \rho^2 \sin \theta \, d\rho \, d\theta \, d\varphi \]

\[ = \iiint f \rho^2 \sin \theta \, d\rho \, d\theta \, d\varphi \]

Spherical coordinates

Ex 2 Find the volume of the solid within the sphere \( x^2 + y^2 + z^2 = 16 \), outside the cone \( z = \sqrt{x^2 + y^2} \) and above the xy-plane.
Jacobian

Let \( x = m(u,v) \) and \( y = n(u,v) \) where \( x, y \) are old variables and \( u, v \) are new variables. I want to change my system from one set of variables to the other.

Define \( J(u,v) \) (the Jacobian) as

\[
J(u,v) = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}
\]

\( = \) \( \iint f(x,y) \, dx \, dy = \iint f[m(u,v), n(u,v)] \, |J(u,v)| \, du \, dv \)

For example, switch from \((x,y)\) to \((r,\theta)\).

\[
x = r \cos \theta \quad y = r \sin \theta
\]

\[
J(r,\theta) = \begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{vmatrix} = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix} = r \cos^2 \theta - (-r \sin^2 \theta) = r \cos^2 \theta + r \sin^2 \theta = r (\sin^2 \theta + \cos^2 \theta) = r
\]
In 3 variables,

\[ J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \]

For example, find \( J(p, \theta, \phi) \) where

\[ x = psin\phi \cos \theta \quad y = psin\phi \sin \theta \quad z = p \cos \phi \]