12.1 Functions of Two or More Variables

Now, we'll deal with functions that take 2 real inputs and give one real output.

E.g., \( f(x, y) = x^2 + 3y^2 \) or \( g(x, y) = \sqrt{xy} + 2x^3 \)

These are real-valued functions of 2 \( \mathbb{R} \) variables.

\[
\begin{align*}
\text{Domain} & \quad \rightarrow \quad \text{Range} \\
(x, y) & \quad \rightarrow \quad 2 = f(x, y)
\end{align*}
\]

Independent variables \( \rightarrow x, y \)
Dependent variable \( \rightarrow z \)

Domain \( \rightarrow \) set of all allowable \( (x, y) \) pairs
Range \( \rightarrow \) set of resulting values

**Example 1**

For \( f(x, y) = \frac{y}{x} + xy \), find \( f(1, 2) \).

Find \( f(a, a) \).

Find \( f\left(\frac{1}{x}, x^2\right) \).

What is the domain of \( f \)?
The graph of a function of 2 variables is a 3D surface (usually). And since it's a function, each output is only one (x, y) from the domain. Graphically, this means that each line 1 to xy-plane intersects the surface in at most one pt.

**Ex 2** Sketch graph of \( f(x, y) = 6 - x^2 \)

**Ex 3** Sketch graph of \( f(x, y) = \sqrt{16 - 4x^2 - y^2} \)
12.1 (continued)

Level Curves = projection of intersecting curves of a surface and planes \( z = c, c \in \mathbb{R} \) onto xy plane.

Contour Map = a collection of level curves

(we are already used to seeing contour maps when dealing w/ temperatures across country or geographical maps.)

\[ \text{You can use computer packages to help you visualize these graphs. Maple + Mathematica are both good} \]
\[ \text{+ available in the computer lab.} \]
\[ \text{(See examples on pg 636-637 in your book!)} \]

Ex 4

Sketch level curves at \( z = -2, -1, 0, 1, 2 \)

for \( \frac{z}{y} = \frac{x}{y} \)
Ex 5. Draw the contour map for 
\[ z = f(x,y) = x^2 + y \quad \text{for} \quad z = -4, -1, 0, 1, 4 \]

For 3 or more variables:
There are lots of cases where we have a function of more than 2 variables. For instance, temperature is dependent on location which is given by 3 coordinates.

Ex 6. Find the domain for 
\[ f(x,y,z) = \sqrt{x^2 + y^2 - z^2 + 9} \]
12.2 Partial Derivatives

Consider the same surface cut by 2 different planes \( \mathbf{i} \) and \( \mathbf{j} \). It's cut by \( y = y_0 \) and \( x = x_0 \). The curve of intersection in \( \mathbf{i} \) goes thru \( RPQ \) and in \( \mathbf{j} \), thru \( MPL \). Each of those curves has a tangent line associated with it at pt \( P \).

Those tangent lines have slopes associated with them and that should make us think about ________!

Since our function is now a fn of 2 variables (rather than 1), we can only take the partial derivative w/r to one of the variables.

\[
fx(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}
\]

slope of tangent live in \( \mathbf{i} \)

\[
fy(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}
\]

slope of tangent live in \( \mathbf{j} \)
Ex 1 Find \( f_x(0,3) \) and \( f_y(0,3) \) if

\[ f(x,y) = 3x^2 y^2 + 4y^3 - 5 \]

Notation

If \( z = f(x,y) \), then

\[
\begin{align*}
f_x(x,y) &= \frac{\partial z}{\partial x} = \frac{df(x,y)}{dx} \quad \text{partial derivative of } f \text{ wrt } x \\
f_y(x,y) &= \frac{\partial z}{\partial y} = \frac{df(x,y)}{dy} \quad \text{partial derivative of } f \text{ wrt } y
\end{align*}
\]

Ex 2 If \( z = x^2 y + \cos(xy) - 2 \), find \( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \).
Ex 3 Find the slope of the tangent to the curve of intersection of the surface $3z = \sqrt{36-9x^2-4y^2}$ and the plane $x=1$ at the point $(1, -2, \sqrt{11}/3)$.

Ex 4 The temperature in degrees Celsius on a metal plate in the xy-plane is given by $T(x,y) = 4 + 2x^2 + y^3$. What is the rate of change of temperature with respect to distance (in ft) if we start moving from $(3, 2)$ in the direction of the y-axis?
Higher Order Partial Derivatives

\[
\begin{align*}
    f_{xx} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \\
    f_{yy} &= \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} \\
    f_{xy} &= (f_x)_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} \\
    f_{yx} &= (f_y)_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}
\end{align*}
\]

called "mixed partials"

Ex 5 Find all 4 second partial derivatives for

(a) \( f(x, y) = (x^3 + y^3)^5 \) \\
(b) \( f(x, y) = \tan^{-1}(xy) \)
Ex 6. For \( f(x,y,z) = xy^2 - \frac{2x}{yz} + 3z^3 x \), find \( f_x, f_y, f_z, f_{xz} \) and \( f_{yz} \).
12.3 Limits and Continuity

Intuitively, \( \lim_{(x,y) \to (a,b)} f(x,y) = L \) means that as the pt \((x,y)\) gets very close to \((a,b)\), then \(f(x,y)\) gets very close to \(L\). When we did this for one-variable functions, it could approach from only 2 sides or directions. Now, however, we can approach \((a,b)\) from any many directions.

**Defn**

\[ \lim_{(x,y) \to (a,b)} f(x,y) = L \text{ means that } \forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \]
\[ 0 < |(x,y) - (a,b)| < \delta \Rightarrow |f(x,y) - L| < \varepsilon. \]

\[ |(x,y) - (a,b)| = \sqrt{(x-a)^2 + (y-b)^2} \]

i.e. the distance from \((x,y)\) to \((a,b)\).

So \(0 < |(x,y) - (a,b)| < \delta\) means all the pts \((x,y)\) inside a circle of radius \(\delta\), centered at \((a,b)\).

(excluding the center, since it has to be bigger than 0)

If different paths of approach lead to different limit values, then the limit does not exist.

**Ex**

\[ \lim_{(x,y) \to (3,1)} [3x^2y - x^3y^2] = 3(3^2)(1) - (3^3)(1^2) \]
\[ = 27 - 27 = 0 \]
Ex 1  Find \( \lim_{(x,y) \to (0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2} \)

Ex 2  Find \( \lim_{(x,y) \to (-2,1)} (xy^3 - xy + 3y^2) \)
12.3 (continued)

Ex 3 Show that \( \lim_{{(x,y) \to (0,0)}} \frac{xy + y^3}{x^2 + y^2} \) does not exist.

Continuity: A fn \( f(x,y) \) is continuous at \( (a,b) \) if
1. the fn exists there,
2. the limit as \( (x,y) \to (a,b) \) exists +
if \( \lim_{{(x,y) \to (a,b)}} f(x,y) = f(a,b) \).
This is basically the same as single-valued functions, if
one variable.

Notice
All polynomial fns have continuity everywhere.
All rational fns are continuous everywhere the denominator
is \( \neq 0 \).

Composite of Fns
If a fn \( g \) of 2 variables is continuous at \( (a,b) \) +
a fn \( f \) of one variable is continuous at \( g(a,b) \), then
\( (f \circ g)(x,y) = f(g(x,y)) \) is continuous at \( (a,b) \).
Ex 4 Show that \( f(x,y) = \sin(x^3 - 4xy) \) is continuous everywhere.

Ex 5 Determine where \( f(x,y) = \ln(1-x^2-y^2) \) is continuous.

Ex 6 Is \( f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases} \) continuous everywhere?
A neighborhood of radius $S$ of pt $P$ is set of all pts $Q$ such that $|Q-P| < S$; i.e., set of pts inside circle centered at $P$ w/ radius $S$.

(In 3d, it means the set of pts inside the sphere centered at $P$ w/ radius $S$.)

A pt $R$ is an interior pt of a set $S$ if there is a neighborhood of $R$ contained in $S$.

B is a boundary pt of $S$ if every neighborhood of $B$ contains pts that are in $S$ and pts not in $S$.

The set of all boundary pts is called the boundary of $S$.

A set is open if all its pts are interior pts.

A set is closed if it contains interior pts as well as all the boundary pts.

**Theorem**: Equality of Mixed Partial Derivatives

If $f_{xy}$ and $f_{yx}$ are continuous on an open set $S$, then $f_{xy} = f_{yx}$ at each pt of $S$.

**Example**: For $S(x,y) = \{ (x,y) : x^2 + y^2 < 4 \}$, sketch set. Describe boundary. Is it open, closed or neither?
12.4 Differentiability

For a function of one variable, the derivative gave the slope of the tangent line. So, for a function of two variables, we should be able to find the tangent plane using the derivative. But the first partial derivatives don't give that info.

For a function of one variable, the tangent line approximates the function for values very close to that point. 
⇒ We say f is almost linear at \( x = a \), where we take the derivative.

\[ f'(a + h) = f(a) + hm + h \varepsilon(h) \]

⇒ locally linear

f is our function

a is x-value where we're finding the slope

h is small change

m is slope of tangent line

\( \varepsilon(h) \) is a function satisfying

\[ \lim_{h \to 0} \varepsilon(h) = 0. \]

(65)
12.4 (continued)

\[ \varepsilon(h) = \frac{f(a+h) - f(a)}{h} \]

slope of second line

\[ \frac{f(a+h) - f(a)}{h} \]

slope of tangent line

If \( f \) is locally linear at \( a \), then

\[ \lim_{{h \to 0}} \varepsilon(h) = \lim_{{h \to 0}} \left( \frac{f(a+h) - f(a)}{h} \right) = 0 \]

\[ \Rightarrow \lim_{{h \to 0}} \frac{f(a+h) - f(a)}{h} = m = f'(a) \]

So, if \( f \) is locally linear, then \( f \) is differentiable.

\[ \Rightarrow \text{if } f \text{ is differentiable, then it's locally linear.} \]

Now, let's see if we can leverage this equivalency for functions of 2 variables.

**Defn.** We say \( f \) is locally linear at \((a,b)\) if

\[ f(a+h_1, b+h_2) = f(a,b) + h_1 f_x(a,b) + h_2 f_y(a,b) \]

\[ + h_1 \varepsilon_1(h_1, h_2) + h_2 \varepsilon_2(h_1, h_2) \]

where \( \varepsilon_1(h_1, h_2) \to 0 \) as \((h_1, h_2) \to 0\) and

\[ \varepsilon_2(h_1, h_2) \to 0 \] as \((h_1, h_2) \to 0\).

Let \( \bar{p}_0 = (a,b) \)

\[ \bar{h} = (h_1, h_2) \]

\[ \varepsilon(h) = (\varepsilon_1(h_1, h_2), \varepsilon_2(h_1, h_2)) \]

Then \( \star \) becomes

\[ f(\bar{p}_0 + \bar{h}) = f(\bar{p}_0) + (f_x(\bar{p}_0), f_y(\bar{p}_0)) \cdot \bar{h} + \varepsilon(\bar{h}) \cdot \bar{h} \]
The function \( f \) is differentiable at \( \overline{p} \) if it is locally linear at \( \overline{p} \). The function \( f \) is differentiable on an open set \( R \) if it is differentiable at every point in \( R \).

**Gradient of \( f \):**

\[
\nabla f(\overline{p}) = (f_x(\overline{p}), f_y(\overline{p})) = f_x(\overline{p})\hat{i} + f_y(\overline{p})\hat{j}
\]

\( (\nabla f(\overline{p})) \) is read "del \( f \) of \( p \)."

\[
\Rightarrow f(\overline{p} + \overline{h}) = f(\overline{p}) + \nabla f(\overline{p}) \cdot \overline{h} + \epsilon(\overline{h}) \cdot \overline{h}
\]

The gradient is the analog of a derivative for a real-valued function! (The gradient can be extended to any dimensions.)

Then if \( f(x,y) \) has continuous partial derivatives \( f_x(x,y) \) and \( f_y(x,y) \) on a disk \( D \) whose interior contains \( (a,b) \), then \( f(x,y) \) is differentiable at \( (a,b) \).

**Properties of \( \nabla \) operator:**

\( \nabla \) is a linear operator so

\( 1 \) \( \nabla [f(\overline{p}) + g(\overline{p})] = \nabla f(\overline{p}) + \nabla g(\overline{p}) \)

\( 2 \) \( \nabla [\alpha f(\overline{p})] = \alpha \nabla f(\overline{p}) \) \( \forall \alpha \in \mathbb{R} \).

Also, \( \nabla [f(\overline{p})g(\overline{p})] = f(\overline{p}) \nabla g(\overline{p}) + g(\overline{p}) \nabla f(\overline{p}) \).

Then if \( f \) is differentiable at \( \overline{p} \), then \( f \) is continuous at \( \overline{p} \).
Ex 1 Find the gradient \( \nabla f \).

(a) \( f(x, y) = x^3 y - y^3 \)

\[
\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}
\]

\[
\frac{\partial f}{\partial x} = 3x^2 y \quad \frac{\partial f}{\partial y} = x^3 - 3y^2
\]

\[
\Rightarrow \quad \nabla f = 3x^2 y \hat{i} + (x^3 - 3y^2) \hat{j}
\]

(b) \( f(x, y) = \sin^3(x^2 y) \)

(c) \( f(x, y, z) = xz \ln(x+y+z) \)
Tangent Plane

If $f$ is differentiable at $\mathbf{p}_0$, then when $h$ has very small magnitude

$$f(\mathbf{p}_0 + h) \approx f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot h$$

Let $\mathbf{p} = \mathbf{p}_0 + h \Rightarrow \mathbf{h} = \mathbf{p} - \mathbf{p}_0$

Then

$$f(\mathbf{p}) \approx f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)$$

So

$$T(\mathbf{p}) = f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)$$

$T(\mathbf{p})$ defines a plane that approximates $f$ near $\mathbf{p}_0$, i.e., the tangent plane

Ex 2: For $f(x, y) = x^3y + 3xy^2$, find the equation of the tangent plane at $\mathbf{p}_0 = (2, -2)$

$$f(2, -2) =$$

$$\nabla f(2, -2) =$$

$$Z = f(2, -2) + \nabla f(2, -2) \cdot \langle x-2, y+2 \rangle =$$
12.4 (continued)

Ex 3 Find eqn of tangent "hyperplane" at $\mathbf{p}_0$.

$$f(x, y, z) = xyz + x^2 \quad \mathbf{p}_0 = (2, 0, -3)$$

Ex 4 Find all pts $(x, y)$ at which tangent plane to graph of $z = x^3$ is horizontal

(Hint: Find tangent plane at $\mathbf{p}_0 = (x_0, y_0)$. Then, you know the normal vector & for a horizontal plane, the normal vector should be $(0, 0, k) \; k \in \mathbb{R}$.)
We know we can write
\[
\frac{\Delta f}{\Delta x} = f_x (\hat{\rho}) = \lim_{h \to 0} \frac{f(\hat{\rho} + h\hat{i}) - f(\hat{\rho})}{h}
\]
\[
\frac{\Delta f}{\Delta y} = f_y (\hat{\rho}) = \lim_{h \to 0} \frac{f(\hat{\rho} + h\hat{j}) - f(\hat{\rho})}{h}
\]

The partial derivatives measure the rate of change of a function in the direction of the x-axis or y-axis. What about rates of change in other directions?

**Defn.** For any unit vector \( \hat{u} \), let
\[
D_u f(\hat{\rho}) = \lim_{h \to 0} \frac{f(\hat{\rho} + h\hat{u}) - f(\hat{\rho})}{h}
\]
If this limit exists, this is called the directional derivative of \( f \) at \( \hat{\rho} \) in the direction of \( \hat{u} \).

Thus let \( f \) be differentiable at \( \hat{\rho} \). Then \( f \) has a directional derivative at \( \hat{\rho} \) in direction of \( \hat{u} \)
\[
\hat{u} = u_1 \hat{i} + u_2 \hat{j} \quad \text{and} \quad D_u f(\hat{\rho}) = \hat{u} \cdot \nabla f(\hat{\rho})
\]
\[
\implies D_u f(x,y) = u_1 f_x(x,y) + u_2 f_y(x,y)
\]
\[
= u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y}
\]
Ex 1  Find the directional derivative of $f$ at $\hat{p}$ in direction $\hat{a}$ (where $\hat{a} = c\hat{u}$)

(a) $f(x, y) = y^2 \ln x$  $\hat{p} = (1, 4)$  $\hat{a} = \hat{i} - \hat{j}$

We first need to make sure we get a unit vector $\hat{u}$ (in direction of $\hat{a}$)

$$\hat{u} = \frac{\hat{a}}{|\hat{a}|} = \frac{\langle 1, -1 \rangle}{\sqrt{1^2 + (-1)^2}} = \frac{\langle 1, -1 \rangle}{\sqrt{2}} = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$$

$$\frac{\partial f}{\partial x} = \frac{y^2}{x} \quad \frac{\partial f}{\partial y} = 2y \ln x \quad \frac{\partial f}{\partial x} \bigg|_{(a, b)} = 16 \quad \frac{\partial f}{\partial y} \bigg|_{(a, b)} = 0$$

$$= \quad D_{\hat{u}} f(x, y) = \frac{\sqrt{2}}{2} (16) + -\frac{\sqrt{2}}{2} (0) = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$$

(b) $f(x, y) = 2x^2 \sin y + yx$  $\hat{p} = (1, \frac{\pi}{2})$  $\hat{a} = 2\hat{a} + \hat{f}$
12.5 (continued)

**Maximum rate of change**

We know $D_{\hat{u}} f(\vec{p}) = \hat{u} \cdot \nabla f(\vec{p})$

$= |\hat{u}| |\nabla f(\vec{p})| \cos \Theta$ where $\Theta$ is angle between $\hat{u} + \nabla f(\vec{p})$

$\Rightarrow D_{\hat{u}} f(\vec{p}) = |\nabla f(\vec{p})| \cos \Theta$ (since $\hat{u}$ is unit vector)

$\Rightarrow D_{\hat{u}} f(\vec{p})$ is max when $\cos \Theta = 1 \Rightarrow \Theta = 0, \pi$.

i.e. when $\hat{u}$ points in direction of gradient.

**Dem**

A fn increases most rapidly at $\vec{p}$ in the direction of the gradient (w/ rate $|\nabla f(\vec{p})|$) and decreases most rapidly in the opposite direction (w/ rate $-|\nabla f(\vec{p})|$).

Pretty cool!

**Ex 2** Find vector in direction of most rapid increase of $f(x, y) = e^y \sin x$ at $\vec{p} = (5\pi/6, 0)$. Then find the rate of change in that direction.
Ex 3. The temperature at \((x, y, z)\) of a ball centered at the origin is 
\[ T(x, y, z) = 100e^{-\left(x^2 + y^2 + z^2\right)} \]
Show that the direction of greatest decrease in temperature is always a vector pointing away from the origin.
12.5 (continued)

One extra (cool) fact

Thus

The gradient of $f$ at pt $P$ is $\perp$ to level curve of $f$ through $P$.

Let's say we have these level curves for our function. Then, $\nabla f$ is always $\perp$ to any pt on the level curves.
12.6 The Chain Rule

Then let \( x = x(t) \) and \( y = y(t) \) be differentiable at \( t \) and let \( z = f(x,y) \) be differentiable at \( (x(t), y(t)) \).
Then, \( z = f(x(t), y(t)) \) is differentiable at \( t \) and
\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]
\( \frac{dz}{dt} = \nabla f \cdot \langle x', y' \rangle \)

Then let \( x = x(s,t) \) and \( y = y(s,t) \) have 1st partial derivatives at \( (s,t) \) and let \( z = f(x,y) \) be differentiable at \( (x(s,t), y(s,t)) \). Then \( z = f(x(s,t), y(s,t)) \) has first partial derivatives given by
\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}
\]
\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}
\]

Ex 1 Find \( \frac{dw}{dt} \) given \( w = x^2 y - y^2 x \), \( x = \cos t \), \( y = \sin t \).
(Express answer in terms of \( t \).)
12.6 (continued)

Ex. 2 Find \( \frac{dw}{dt} \) given \( w = \ln(x+y) - \ln(xy) \)
\[x = te^s, \quad y = e^{st}\] (Express answer in \( s \) and \( t \)).

Ex. 3 Find
\[
\frac{\partial z}{\partial s} \bigg|_{r=1, s=-1, t=2}
\]
If \( z = xy + x + y \) \quad \[x = r + s + t \quad y = rst,\]
Implicit Differentiation

Let's go back to \( y = f(x) \) for a moment, and assume that instead of getting \( y \) as a function of \( x \) (explicitly), we have \( F(x, y) = 0 \) (i.e. \( y \) is defined implicitly).

Then, we just differentiate both sides \( \text{wrt} \ x \) to get

\[
\frac{dy}{dx} \quad \frac{d}{dx}(y^3 - 2xy + 3x) = \frac{d}{dx}(4)
\]

\[
3y^2 \frac{dy}{dx} - (2x \frac{dy}{dx} + 2y) + 3 = 0
\]

\[
\frac{dy}{dx} (3y^2 - 2x) = 2y - 3
\]

\[
\frac{dy}{dx} = \frac{2y - 3}{3y^2 - 2x}
\]

Now that we have partial derivatives, we could think of this process as

\[
F(x, y) = 0
\]

\[
\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0 \quad + \text{solve for} \quad \frac{dy}{dx}
\]

knowing that \( \frac{dx}{dx} = 1 \)

\[
=) \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0 \quad \iff \quad \frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}
\]

\( \text{Chain Rule} \)
If we expand this thinking to a function of two variables, we get

\[
\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} \quad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}
\]

where \( F(x,y,z) = 0 \) is the beginning eqn.

Ex 4 If \( ye^x + 8\sin x = 0 \), find \( \frac{\partial x}{\partial y} \).
12.7 Tangent Planes

We already dealt with tangent planes (in 15.4) to surfaces of form \( z = f(x,y) \). Now, we'll do tangent planes to surfaces of form \( F(x,y,z) = 0 \), i.e., a surface represented by any equation in 3 variables.

If \( x = x(t), y = y(t), z = z(t), \) for \( t \), then

\[ F(x(t), y(t), z(t)) = k \]

is our generic surface.

\[
\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = \frac{dk}{dt} = 0
\]

\[
\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle = 0
\]

\[
\nabla F \cdot \frac{d\vec{r}}{dt} = 0 \quad \text{where} \quad \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}
\]

and remember that \( \frac{d\vec{r}}{dt} \) is vector in direction of tangent line to the curve \( \Rightarrow \) the gradient \( \bot \) to tangent line at \( pt \) to \( \vec{r}(t) \).

**Defn**

Let \( F(x,y,z) = k \) be a surface, \( F \) differentiable at \( P(x_0,y_0,z_0) \) and \( \nabla F(x_0,y_0,z_0) \neq 0 \). Then the plane through \( P \) \( \bot \) to \( \nabla F(x_0,y_0,z_0) \) is called the tangent plane to the surface at \( P \).
Thus
For surface \( F(x, y, z) = k \), eqn of tangent plane at \((x_0, y_0, z_0)\) is
\( \nabla F(x_0, y_0, z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0 \)
\[ \Rightarrow F_x(x_0, y_0, z_0)(x-x_0) + F_y(x_0, y_0, z_0)(y-y_0) + F_z(x_0, y_0, z_0)(z-z_0) = 0. \]

This is because \( P \) is a pt on the plane and \( \nabla F \) is the normal. So, by the defn of a plane eqn, we get the above result.

**EX 1** Find the eqn of the tangent plane to \( 8x^2 + y^2 + 8z^2 = 16 \) at \((1, 1, 1/2)\).
Ex. 2 Find the parametric equs of the line that is tangent to the curve of intersection of the surfaces $f(x,y,z) = 9x^2 + 4y^2 + 4z^2 - 41 = 0$ and $g(x,y,z) = 2x^2 - y^2 + 3z^2 - 10 = 0$ at the pt $(1, 2, 2)$. 
12.7 (continued) (extension of 2.9 on differentials + approximations)

Defn
Let \( z = f(x,y) \), \( f \) is differentiable in \( dx + dy \)
(differentials) are variables. \( dz \) (also called total differential of \( f \)) is

\[
dz = df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy = \nabla f \cdot <dx,dy>
\]

Ex 3 Use \( dz \) to approximate change in \( z \) as \( (x,y) \) moves from \( P \) to \( Q \). Also, find \( \Delta z \).

\( z = x^2 - 5xy + y \)  \( P (2,3) \)  \( Q (2.03, 2.98) \)
Taylor Polynomial for functions of 2 variables

\[ P_2(x,y) = f(x_0, y_0) + \left[ f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) \right] \]
\[ + \frac{1}{2} \left[ f_{xx}(x_0, y_0)(x-x_0)^2 + 2f_{xy}(x_0, y_0)(x-x_0)(y-y_0) \right. \]
\[ \left. + f_{yy}(x_0, y_0)(y-y_0)^2 \right] \]

(2nd order Taylor polynomial, centered at \((x_0, y_0)\))

Ex 4: For \( f(x,y) = \tan\left( \frac{x^2+y^2}{6y} \right) \) find 2nd order Taylor polynomial based at \((0,0)\). Then estimate \( f(0.2, -0.3) \), using
1. Taylor polynomial
2. Calculator.
12.8 Maxima & Minima

**Theorem (Max-Min Existence)**

If $f$ is continuous on a closed, bounded set $S$, then $f$ attains both a global max value and global min value there.

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**Theorem (Critical Pt Thm)**

Let $f$ be defined on a set $S$ containing $\bar{p}$. If $f(\bar{p})$ is an extreme value, then $\bar{p}$ must be a critical pt, i.e. either $\bar{p}$ is

1. a boundary pt of $S$

or 2. a stationary pt of $S$ (pt $\bar{p}$ where $Df(\bar{p}) = 0$)

or 3. a singular pt of $S$, (a pt where $f$ is not differentiable)
Second Partial Test Theorem

Suppose $f(x,y)$ has continuous second partial derivatives in neighborhood of $(x_0, y_0)$ and $f(x_0, y_0) = 0$. Let

$$D = D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$$

Then

1. If $D > 0$, $f_{xx}(x_0, y_0) < 0$, $f(x_0, y_0)$ local max
2. If $D > 0$, $f_{xx}(x_0, y_0) > 0$, $f(x_0, y_0)$ local min
3. If $D < 0$, $f(x_0, y_0)$ not an extreme value
   $(x_0, y_0)$ is saddle pt.
4. If $D = 0$, test is inconclusive.

Example 1: For $f(x,y) = xy^2 - 6x^2 - 3y^2$, find all critical pts.

Indicate whether each is min, max or saddle pt.
Ex 2. Find global max value + min value of 
\[ f(x,y) = x^2 + y^2 \] on 
\[ S = \{(x,y) \mid x \in [1,3], y \in [1,4]\} \]

pts that yield those min + max values.

\[ z = x^2 + y^2 \]

is right circular cone
Ex 3 Find global max and min pts for
\[ f(x,y) = x^2 - 6x + y^2 - 8y + 7 \] on \( S = \{(x,y) \mid x^2 + y^2 \leq 1\} \)
12.8 (continued)

Ex 4 Find the 3d vector of length 9 +
whose sum of its components is a max.
12.9 Lagrange's Method

We saw (in last section) examples of constrained extremum problems, i.e. max or min problems with some condition/constraint. Now, we'll see easier way to solve those.

Optimize $f(x,y)$ subject to constraint $g(x,y) = 0$.

Graphically, let curves be level curves of $f(x,y)$, i.e. where $f(x,y) = k$, $k$ constant. And curve is constraint curve.

To maximize $f$ subject to $g(x,y) = 0$ means to find the level curve of $f$ w/ greatest $k$-value that intersects constraint curve. It will be a place where two curves are tangent! (Likewise for the minimum value of $f$.)

$\Rightarrow$ Two curves have a common tangent line (if they're tangent at that pt)

And, we know the If is $\perp$ to its level curves! $\nabla g$ is also $\perp$ to constraint curve.
12.9 (continued)

\[ \nabla f + \lambda \nabla g = \lambda \nabla f + \nabla g \]

\[ \nabla f(\vec{p}_0) = \lambda_0 \nabla g(\vec{p}_0) \quad \text{and} \quad \nabla f(\vec{p}_1) = \lambda_1 \nabla g(\vec{p}_1) \]

\[ \lambda_0, \lambda_1 \in \mathbb{R} \quad \lambda_0 \neq 0, \lambda_1 \neq 0. \]

**The Lagrange's Method**

To maximize or minimize \( f(\vec{p}) \) subject to constraint \( g(\vec{p}) = 0 \), solve system of eqns

\[ \nabla f(\vec{p}) = \lambda \nabla g(\vec{p}) \quad \text{and} \quad g(\vec{p}) = 0 \]

for \( \vec{p} \) and \( \lambda \). Each pt \( \vec{p} \) is a critical pt for constrained extremum problem + corresponding \( \lambda \) is called Lagrange multiplier.

**Ex 1** Find max of \( f(x,y) = xy \) subject to constraint \( g(x,y) = 4x^2 + 9y^2 - 36 = 0 \).
Ex 2  Find the least distance between the origin and the plane $x + 3y - 2z = 4$. 

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12.9 (continued)

If we have more than one constraint, additional Lagrange multipliers are used. If we want to maximize \( f(x,y,z) \) subject to \( g(x,y,z) = 0 \) and \( h(x,y,z) = 0 \), then we solve

\[
\nabla f = \lambda \nabla g + \mu \nabla h \quad g = 0 \quad h = 0.
\]

**Ex 3** Find the max volume of the 1st-octant rect. box (w/ faces || to coordinate planes) w/ one vertex at \((0,0,0)\) & diagonally opposite vertex on plane \(3x - y + 2z = 1\)
12.9 (continued)

Example 4: Find the minimum distance from the origin to the line of intersection of the two planes:

\[ x + y + z = 8 \quad \text{and} \quad 2x - y + 3z = 28. \]

Let

\[ f(x, y, z) = x^2 + y^2 + z^2 \]

\[ g(x, y, z) = x + y + z - 8 = 0 \]

\[ h(x, y, z) = 2x - y + 3z - 28 = 0 \]