10.4 Plane Curves: Parametric Representation

Remember from 5.4 (in Math2210), a plane curve is a 2d curve given by

\[ x = f(t) \quad y = g(t) \quad \text{w/ } t \in I \]

+ \( f + g \) are continuous functions on interval \( I = [a, b] \)
+ \( t \) = parameter (we can think of it measuring time)

\[ t \in [a, b] \]

\[ P \rightarrow Q(x(b), y(b)) \]
\[ \text{initial endpoint} \]
\[ x(a), y(a) \]
\[ \text{final endpoint} \]

Simple curve

Not closed

Arrowheads indicate \( t \) going from \( a \) to \( b \).

\[ P = Q \]
\[ \text{simple, closed curve} \]

\[ P = Q \]
\[ \text{closed, not simple curve} \]

\[ P = Q \]
\[ \text{not simple, not closed curve} \]

It's harder to recognize curve shape when given parametrically. Sometimes, it's possible to eliminate the parameter.
Ex 1 \[ x = t - 3 \quad y = \sqrt{2t} \quad 0 \leq t \leq 8 \]

\[ y^2 = 2t \]

\[ \frac{y^2}{2} = t \]

\[ \Rightarrow x = \frac{y^2}{2} - 3 \]

which we know is a parabola (facing right)

But, we only want that piece of the sideways parabola when \( 0 \leq t \leq 8 \)

\[ \Rightarrow -3 \leq x \leq 5 \quad \text{and} \quad 0 \leq y \leq 4 \]

This is a simple, not closed curve.

Ex 2 For \( x = 3\sqrt{t} - 3 \quad y = 2\sqrt{4-t} \quad 3 \leq t \leq 4 \)

eliminate the parameter \( t \), graph the curve and tell if it's simple and closed.
Ex 3  For \( x = \sin \theta \) and \( y = 2\cos^2(2\theta) \) \( \theta \in \mathbb{R} \),
eliminate the parameter \( \theta \)
and graph the curve and tell if it's simple or closed.
10.4 (continued)

**Theorem A**

Let $f + g$ be continuously differentiable on $t \in (a, b)$. Then, the parametric equations

$$x = f(t), \quad y = g(t)$$

define $y$ as a differentiable function of $x$ if

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

where

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy/dt}{dx/dt} \right) = \frac{dy}{dx} \left( \frac{dx}{dt} \right)^{-1}$$

**Exercise 4**

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (w/o eliminating parameter)

(a) $x = \sqrt{3} \theta^2, \quad y = -\sqrt{3} \theta^3 \quad \theta \neq 0$

(b) $x = \frac{2}{1+t^2}, \quad y = \frac{2}{t(1+t^2)} \quad t \neq 0$
10.4 (continued)

Ex 5

Find length of curve given by

\[ x = \sin \theta - \theta \cos \theta \]
\[ y = \cos \theta + \theta \sin \theta \]

Note: \[ L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]
arc-length formula (Secn 6.4)
11.1 Cartesian Coordinates in 3-Space

A point in 3-space is given by an ordered triple \((x, y, z)\).

This is a right-handed system (because if I curl my fingers from the \(x\)-direct to the \(y\)-direct, then my thumb points in the \(z\)-direct).

* A right-handed system is standard.

\[ \text{Plot } (2, 3, 4) \]
\[ \text{Plot } (-1, 4, -3) \]

**Distance Formula**

\[ \Delta P_{1}P_{2}Q + \Delta P_{1}QR \text{ are both right } \Delta s. \]
\[ \Rightarrow |P_{1}Q|^{2} + |Q_{1}P_{2}|^{2} = |P_{1}P_{2}|^{2} \]
\[ \text{and } |P_{1}R|^{2} + |Q_{1}R|^{2} = |P_{1}R|^{2} \]
by Pythagorean Thm.
\[ \Rightarrow |P_{1}R|^{2} + |RQ|^{2} + |Q_{1}P_{2}|^{2} = |P_{1}P_{2}|^{2} \]

6
11.1 (continued)

If \( P_1 = (x_1, y_1, z_1) \) and \( P_2 = (x_2, y_2, z_2) \), then

\[ R = (x_2, y_1, z_1) \] and \( Q = (x_1, y_2, z_1) \).

\[ \Rightarrow |P_1 R|^2 = (\sqrt{(x_2-x_1)^2})^2 = (x_2-x_1)^2 \]

and likewise \( |RQ|^2 = (y_2-y_1)^2 \), \( |QP_2|^2 = (z_2-z_1)^2 \)

\[ \Rightarrow |P_1 P_2|^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 \]

\[ \Rightarrow |P_1 P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \]

distance between \( P_1 \) and \( P_2 \)

Ex 1. Show that \((4, 5, 3), (1, 7, 4)\) and \((2, 4, 6)\) are vertices of an equilateral triangle.
Spheres

For a sphere, all pts (x,y,z) on surface of sphere are a fixed distance, r, from its center.

That is,

\[ r = \sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2} \]

\[ \Rightarrow r^2 = (x-h)^2 + (y-k)^2 + (z-l)^2 \]

Sphere Eqn.

If anything fits this formula, then it's either a sphere (if \( r > 0 \)), a single pt (if \( r = 0 \)) or the empty set (if \( r < 0 \)).

Ex 2 Given \( x^2 + y^2 + z^2 + 2x - 6y - 10z + 34 = 0 \), find the center & radius of this sphere.
11.1 (continued)

Midpt \( M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right) \) is midpt
between \((x_1, y_1, z_1) + (x_2, y_2, z_2)\).

Linear Eqns (in 3-space)
a linear eqn of the form \( Ax + By + Cz = D \)
\( A^2 + B^2 + C^2 \neq 0 \) graphs into a plane

Ex 3  Graph  \( 3x - 4y + 2z = 24 \)
Get x-intercept:
y-intercept:
z-intercept:

Ex 4  Graph  \( 3x + 4y = 12 \)

A trace = line of intersection between plane w/ coordinate planes
11.2 Vectors (Geometric Approach)

Scalars $\Rightarrow \mathbb{R}$ numbers

Vector $\Rightarrow$ has $\Box$ direction $+ \Box$ magnitude; in book, they show up as boldfaced lowercase letters, like $\mathbf{u}$; in writing, its notation is $\vec{u}$.

Magnitude $\Rightarrow$ length of vector; denoted by $|\vec{u}|$.

Direction $\Rightarrow$ of vector is geometrically indicated by an arrowhead.

\[ \vec{u} = \vec{v} \text{ if magnitude + direction of } \vec{u} + \vec{v} \text{ are the same.} \]

Notice that location doesn't matter for a vector, only magnitude (a.k.a. length) + direction.

Zero vector $\Rightarrow \vec{0}$ and $\vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$

(a zero vector has zero magnitude)

$-\vec{u}$ $\Rightarrow$ means same vector as $\vec{u}$, only pointing in opposite direction.

Scalar multiple of $\vec{u} = \Rightarrow c\vec{u}$, where $c \in \mathbb{R}$, means we have a vector in the direction of $\vec{u}$ but scaled in length.

\[ \vec{u} \quad 2\vec{u} \quad \frac{1}{2}\vec{u} \]
Adding vectors: $\vec{u} + \vec{v}$

Put $\vec{u} + \vec{v}$ so tails are coincident. Then, they form 2 sides of a parallelogram. And $\vec{u} + \vec{v}$ is diagonal of that parallelogram with its tail at same location as tails of $\vec{u} + \vec{v}$.

Place $\vec{u} + \vec{v}$ tail-to-tail. Then, $\vec{u} + \vec{v}$ initiates at tail of $\vec{u}$ and ends at head of $\vec{v}$.

Ex 1

Express $\vec{w}$ in terms of $\vec{u} + \vec{v}$.

Ex 2

Draw $\vec{w}$ as $\vec{w} = \vec{u}_1 + \vec{u}_2 + \vec{u}_3$
Ex 3 (#10) Mark pushes on a post in the direction S 30° E (30° East of South), with a force of 60 lbs. Dan pushes on the same post in the direction S 60° W with a force of 80 lbs. What are the magnitude and direction of the resulting force?

\[ \mathbf{w} = \mathbf{u} + \mathbf{v} \] as drawn
Ex 4 (#14)

A ship is sailing due south at 20 mph. A man walks west across the deck at 3 mph. What are the magnitude and direction of his velocity relative to the surface of the water?
11.2 Vectors (Algebraic Approach)

If we place our vector \( \vec{u} \) on a Cartesian coordinate system with its tail at the origin \((0,0,0)\), then its head will end at some point \((u_1,u_2,u_3)\). Then, we say \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) (Notice the different brackets which distinguish between pts + vectors!)

\( u_1, u_2, u_3 \) are called components of \( \vec{u} \).

\( \vec{u} + \vec{v} \) are equal if and only if \( u_1 = v_1, u_2 = v_2, u_3 = v_3 \)

\[ \vec{u} + \vec{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \]

\[ -\vec{u} = \langle -u_1, -u_2, -u_3 \rangle \]

\[ c\vec{u} = \langle cu_1, cu_2, cu_3 \rangle \]

\[ \vec{0} = 0\vec{u} = \langle 0, 0, 0 \rangle \]

**Theorem A**

For any vectors \( \vec{u}, \vec{v}, \vec{w} \) and \( a, b \in \mathbb{R} \),

\[ \vec{u} + \vec{v} = \vec{v} + \vec{u} \]
\[ (\vec{a} + \vec{b}) + \vec{w} = \vec{a} + (\vec{v} + \vec{w}) \]
\[ \vec{u} + \vec{0} = \vec{u} \]
\[ \vec{u} + (-\vec{u}) = \vec{0} \]
\[ a(b\vec{u}) = (ab)\vec{u} = \vec{u}(ab) \]
\[ a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v} \]
\[ (a+b)\vec{u} = a\vec{u} + b\vec{v} \]
\[ \vec{0} = \vec{u} \]

i.e., vector addition is commutative + associative. 
To still have an additive identity element + additive inverses. 
3. Scalar multiplication is commutative, associative + distributive, and 4. there's a scalar multiplicative identity! We get to play a cool new algebra game with vectors now!!
11.2 (continued)

\[ |\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} \] (Pythagorean Theorem disguised again)

\[ |c\vec{u}| = |c||\vec{u}| \quad |c| \text{ is abs. value of } c \]
but \[ |\vec{u}| \text{ is magnitude of vector } \vec{u} \]

Ex 5 let \( \vec{u} = \langle 1, 5, 2 \rangle \). Find \( |\vec{u}| + |1-3\vec{u}| \).
Also, find a vector \( \hat{u} \) (w/ same direction as \( \vec{u} \) but with length of 1).

\[
\begin{align*}
\hat{u} &= \text{unit vector of } \vec{u} \\
|\hat{u}| &= 1 \\
\hat{u} &= \frac{\vec{u}}{|\vec{u}|}
\end{align*}
\]

Basis vectors = \( \vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle \) and \( \vec{k} = \langle 0, 0, 1 \rangle \).
All three are unit vectors in \( \hat{x}, \hat{y}, \hat{z} \) directions, respectively.

\( \Rightarrow \) \( \vec{u} = \langle u_1, u_2, u_3 \rangle = u_1\vec{i} + u_2\vec{j} + u_3\vec{k} \) (just different notation for same vector)
11.3 The Dot Product

The dot product $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$ (aka scalar product).

**Theorem A**

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors, $c \in \mathbb{R}$.

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutative)
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive over addition/subtraction)
- $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ (scalar mult. with dot product is associative and commutative)
- $\mathbf{0} \cdot \mathbf{u} = 0$
- $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$

**Theorem B**

Geometrically, we can think of dot product as $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta$ where $\theta$ is smallest nonnegative angle between $\mathbf{u}$ and $\mathbf{v}$.

We can use this $\Delta$ made by the drawn vectors. Apply Law of Cosines,

$$||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2 ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta.$$ 

Also, we know (from dot product properties) that

$$||\mathbf{u} - \mathbf{v}||^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$$

$$= \mathbf{u} \cdot (\mathbf{u} - \mathbf{v}) - \mathbf{v} \cdot (\mathbf{u} - \mathbf{v})$$

$$= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}$$

$$= ||\mathbf{u}||^2 - 2 \mathbf{u} \cdot \mathbf{v} + ||\mathbf{v}||^2$$

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So \[ |\vec{u} - \vec{v}|^2 = |\vec{u}|^2 - \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} + |\vec{v}|^2 \]
\[ = |\vec{u}|^2 + |\vec{v}|^2 - 2 \vec{u} \cdot \vec{v} \quad \text{(B)} \]

\[ \Rightarrow \text{ (Equate \ (A) + (B))} \]
\[ |\vec{u}|^2 + |\vec{v}|^2 - 2 |\vec{u}| |\vec{v}| \cos \theta = |\vec{u}|^2 + |\vec{v}|^2 - 2 \vec{u} \cdot \vec{v} \]
\[ -2 |\vec{u}| |\vec{v}| \cos \theta = -2 \vec{u} \cdot \vec{v} \]
\[ |\vec{u}| |\vec{v}| \cos \theta = \vec{u} \cdot \vec{v} \]

Then \[ \vec{u} + \vec{v} \] are \( \perp \text{ iff } \vec{u} \cdot \vec{v} = 0 \]

Ex. 1. For what \( c \) are \( <2c, -8, 17> \perp <3, c, c - 2> \)?
11.3 (cont)

Ex 2 (a) Write vector represented by \( \vec{AB} \) in the form \( \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \).

\[ A(-2,3,5) \quad B(1,-2,4) \]

(b) Find a unit vector \( \vec{u} \) in the direction of \( <-3,5,6> \) and express it as \( \vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k} \).
\[ |\hat{u}| = |\hat{v}| \cos \alpha = \hat{u} \cdot \hat{v} \iff |\hat{u}| \cos \alpha = \langle u_1, u_2, u_3 \rangle \cdot \langle 0, 0, 0 \rangle \]
\[ |\hat{u}| \cos \alpha = u_1, \]

\[
\cos \alpha = \frac{u_1}{|\hat{u}|} \quad \text{and} \quad \cos \beta = \frac{u_2}{|\hat{u}|}, \quad \cos \gamma = \frac{u_3}{|\hat{u}|}
\]

where
- \(\alpha\) = angle between vector \(\hat{u}\) + x-axis
- \(\beta\) = angle between vector \(\hat{u}\) + y-axis
- \(\gamma\) = angle between vector \(\hat{u}\) + z-axis

Notice

\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \]

\[
\left( \frac{u_1}{|\hat{u}|} \right)^2 + \left( \frac{u_2}{|\hat{u}|} \right)^2 + \left( \frac{u_3}{|\hat{u}|} \right)^2
\]

\[
= \frac{u_1^2 + u_2^2 + u_3^2}{|\hat{u}|^2} = \frac{|\hat{u}|^2}{|\hat{u}|^2} = 1
\]

i.e.

\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1
\]
Ex 3 Find the direction cosines for
\[ \vec{u} = \langle -1, 2, -2 \rangle. \]

Ex 4 Find the angle between \( -4\vec{i} + 2\vec{j} + 3\vec{k} = \vec{u} \) and \( \vec{v} = 2\vec{i} + 8\vec{j} + 5\vec{k} \).
11.3 (continued)

\[ \mathbf{w} = \mathbf{u} \cos \theta \]

Since we know \( \cos \theta = \frac{|\mathbf{w}|}{|\mathbf{u}|} \) \( \Rightarrow \) \( |\mathbf{w}| = |\mathbf{u}| \cos \theta \)

But \( \mathbf{w} \) is in the same direction as \( \mathbf{v} \), so
\( \mathbf{w} = c \mathbf{v} \) must be true for some \( c \in \mathbb{R} \)

\( \Rightarrow \) \( |\mathbf{w}| = c |\mathbf{v}| \) combine with 0 above

\( \Rightarrow \) \( |\mathbf{u}| \cos \theta = c |\mathbf{v}| \) \( \Rightarrow \) \( c = \frac{|\mathbf{u}|}{|\mathbf{v}|} \cos \theta \)

But \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \),
\( \Rightarrow \) \( \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta \) \( \Leftrightarrow \) \( \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \)

\( \Rightarrow \) \( c = \frac{|\mathbf{u}|}{|\mathbf{v}|} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \)

\( \Rightarrow \) \( \mathbf{w} = c \mathbf{v} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \)

This is the projection of \( \mathbf{u} \) onto \( \mathbf{v} \)!

Notation: \( \text{proj}_{\mathbf{v}} \mathbf{u} = \text{projection of } \mathbf{u} \text{ onto } \mathbf{v} \)
11.3 (cont)

Ex 5 Let \( \vec{u} = \langle 1, 6, -2 \rangle + \vec{v} = \langle -3, 2, 5 \rangle \).
Find the vector projection of \( \vec{u} \) onto \( \vec{v} \).

Ex 6 If \( \vec{u} = e^1 + \pi \vec{j} + \vec{k} + \vec{v} = \langle 1, 1, 0 \rangle \), express \( \vec{u} \) as the sum of a vector \( \vec{m} \parallel \vec{v} \) and \( \vec{n} \perp \vec{v} \).
\( \vec{m} = \rho \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} \) and \( \vec{n} = \vec{u} - \vec{m} \)
Planes \( \vec{n} = \langle A, B, C \rangle \) If we have a plane with a nonzero normal \( \vec{n} = \langle A, B, C \rangle \), then every pt \( P(x, y, z) \) will satisfy \( \overrightarrow{P_1P} \cdot \vec{n} = 0 \) where \( P_1(x_1, y_1, z_1) \) is a pt on the plane (and every \( P \) is on the plane).

\[ \overrightarrow{P_1P} = \langle x-x_1, y-y_1, z-z_1 \rangle \]

and \( \overrightarrow{P_1P} \cdot \vec{n} = \langle x-x_1, y-y_1, z-z_1 \rangle \cdot \langle A, B, C \rangle \]

\[ = A(x-x_1) + B(y-y_1) + C(z-z_1) = 0 \]

---

Standard eqn of a plane

Generally, a plane is given by a linear eqn \( Ax + By + Cz = D \) \( \Rightarrow A^2 + B^2 + C^2 \neq 0 \) (i.e., the normal vector can't be the zero vector).

**Ex 4.** Find the eqn of the plane through \((1, 3, 4)\) \& \( \vec{n} = \langle 1, 2, -1 \rangle \).
Distance from \( p_1 (x_0, y_0, z_0) \) to plane \( Ax + By + Cz = D \Rightarrow \) We want the distance from \( p_1 \) to the plane. So we want to project \( \vec{m} \) onto \( \vec{n} \) and find that distance.

Let \( \vec{m} = \) vector from \( p_1 \) to \( (x_0, y_0, z_0) \). 

\[ L = \left| \frac{\vec{m} \cdot \vec{n}}{|\vec{n}|} \right| \]

\[ L = \frac{|m - n|}{|n|} \]

\[ \vec{n} = \langle A, B, C \rangle \]

\[ \vec{m} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle \]

\[ L = \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{\sqrt{A^2 + B^2 + C^2}} \]

\[ L = \frac{|(Ax_0 + By_0 + Cz_0) - (Ax_1 + By_1 + Cz_1)|}{\sqrt{A^2 + B^2 + C^2}} \]

But \( (x_1, y_1, z_1) \) is on the plane \( Ax + By + Cz = D \)

\[ \Rightarrow L = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} \]

Distance from \( p_1 \) \( (x_0, y_0, z_0) \) to plane \( Ax + By + Cz = D \)

\[ (x_0, y_0, z_0) \text{ is random pt in space} \]

\[ p_1 \text{ in space} \]

\( \bigstar \) see next "add-on" page 24 notes for explanation.
\[ L = |\text{proj}_\mathbf{n} \mathbf{m}| \]

\[ \text{proj}_\mathbf{n} \mathbf{m} = \left( \frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{n}|^2} \right) \mathbf{n} \]

\[ \Rightarrow |\text{proj}_\mathbf{n} \mathbf{m}| = \left| \left( \frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{n}|^2} \right) \mathbf{n} \right| = \frac{|\mathbf{m} \cdot \mathbf{n}|}{|\mathbf{n}|^2} = \frac{|\mathbf{m} \cdot \mathbf{n}|}{|\mathbf{n}|^2} \\
\]

\[ |\text{proj}_\mathbf{n} \mathbf{m}| = \left| \frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{n}|} \right| = L \]

Using trig (rather than projection)

\[ \Rightarrow \mathbf{L} = |\mathbf{m}| \cos \theta \]

\[ \text{but } \cos \theta = \frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{m}| |\mathbf{n}|} \Rightarrow L = |\mathbf{m}| \left( \frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{m}| |\mathbf{n}|} \right) \quad \text{where we assume we've chosen } \theta \text{ as the } \text{"smallest" angle between } \mathbf{m} \text{ and } \mathbf{n} \Rightarrow \cos \theta \text{ is positive} \]

\[ \mathbf{n} \perp \mathbf{m} \]
Ex. 5 Find the distance between parallel planes
\[-3x + 2y + z = 9 \quad \text{and} \quad 6x - 4y - 2z = 19.\]

Ex. 6 Find the (smallest) angle between two planes
\[3x - 2y + 5z = 7 \quad \text{and} \quad 4x - 2y - 3z = 2.\]
11.4 The Cross Product

\[ \vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle \]

where \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) and \( \vec{v} = \langle v_1, v_2, v_3 \rangle \)

Notice that a dot product yields a scalar, but a cross product yields a vector!

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix}
= \hat{i}(u_2v_3 - u_3v_2) + \hat{j}(u_3v_1 - u_1v_3) + \hat{k}(u_1v_2 - u_2v_1)
\]

This format is easier to remember than the above formula. You just need to remember determinants.

Notice \( \vec{u} \times \vec{v} = -\vec{v} \times \vec{u} \)

Ex 1: If \( \vec{a} = \langle 3, 3, 17 \rangle \) and \( \vec{b} = \langle -2, -1, 0 \rangle \) and \( \vec{c} = \langle -2, 3, 17 \rangle \), find \( \vec{a} \times (\vec{b} \times \vec{c}) \)
11.4 (continued)

**Thm A**

Let $\mathbf{\hat{u}} + \mathbf{\hat{v}}$ be 3d vectors and $\theta$ is angle between them. Then

1. $\mathbf{\hat{u}} \cdot (\mathbf{\hat{u}} \times \mathbf{\hat{v}}) = 0 = \mathbf{\hat{v}} \cdot (\mathbf{\hat{u}} \times \mathbf{\hat{v}})$, i.e. $\mathbf{\hat{u}} \times \mathbf{\hat{v}}$ is 90° to both $\mathbf{\hat{u}}$ and $\mathbf{\hat{v}}$.

2. $\mathbf{\hat{u}}, \mathbf{\hat{v}} + \mathbf{\hat{u}} \times \mathbf{\hat{v}}$ form a right-handed triple.

3. $|\mathbf{\hat{u}} \times \mathbf{\hat{v}}| = |\mathbf{\hat{u}}| |\mathbf{\hat{v}}| \sin \theta$

**Proof**

1. $\mathbf{\hat{u}} \cdot (\mathbf{\hat{u}} \times \mathbf{\hat{v}}) = \langle u_1, u_2, u_3 \rangle \cdot \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$

   $= u_1 (u_2 v_3 - u_3 v_2) + u_2 (u_3 v_1 - u_1 v_3) + u_3 (u_1 v_2 - u_2 v_1)$

   $= u_1 u_2 v_3 - u_1 u_3 v_2 + u_2 u_3 v_1 - u_2 u_1 v_3 + u_3 u_1 v_2 - u_3 u_2 v_1$

   $= 0$. \(\Box\)

**Thm B**

Two 3d vectors $\mathbf{\hat{u}} + \mathbf{\hat{v}}$ are parallel (par) iff $\mathbf{\hat{u}} \times \mathbf{\hat{v}} = \mathbf{0}$

**Ex 2** Find the plane through 3 pts $(-1, 3, 0)$, $(5, 1, 2)$, and $(4, -3, -1)$.
Ex 3 Find area of parallelogram \( \vec{a} + \vec{b} \) as adjacent sides.

\[ \sin \theta = \frac{h}{|\vec{a}|} \Rightarrow h = |\vec{a}| \sin \theta \]

Ex 4 Find volume of parallelogram prism (box), determined by sides \( \vec{a}, \vec{b}, \vec{c} \).

Parallelogram base area is \( \vec{b} \times \vec{c} \)

\[ V = \text{area of base} \times \text{ht} \]

\[ \cos \theta = \frac{h}{|\vec{a}|} \Rightarrow h = |\vec{a}| \cos \theta \quad \text{but} \quad \cos \theta = \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{a}| |\vec{b} \times \vec{c}|} \]
Thm C

\( \vec{u}, \vec{v}, \vec{w} \) are 3d vectors, \( k \in \mathbb{R} \):

1. \( \vec{u} \times \vec{v} = - (\vec{v} \times \vec{u}) \) \hspace{1cm} (Anticommutativity)
2. \( \vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w}) \) \hspace{1cm} (Left distributivity)
3. \( k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v}) \)
4. \( \vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0} \) \hspace{1cm} \( \vec{u} \times \vec{u} = \vec{0} \)
5. \( (\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w}) \)
6. \( \vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w} \)

\[ \hat{i} \times \hat{j} = \hat{k} \hspace{1cm} \hat{j} \times \hat{k} = \hat{i} \hspace{1cm} \hat{k} \times \hat{i} = \hat{j} \]

Ex 5 Calculate \( \vec{u} \times \vec{v} \) if \( \vec{u} = 2\hat{i} - 3\hat{j} + \hat{k} \) and \( \vec{v} = -5\hat{i} + 4\hat{j} \).
\textbf{11.5 Vector-Valued Functions + Curvilinear Motion}

A \underline{vector-valued function} is a function $F : t \mapsto \mathbb{R}^n$ that associates every input $t$ with an output vector $\vec{F}(t)$.

\[ \vec{F}(t) = f(t)\hat{i} + g(t)\hat{j} = \langle f(t), g(t) \rangle \]

where $f$ and $g$ are real-valued functions of $t$.

**Definition**

\[ \lim_{t \to c} \vec{F}(t) = \vec{L} \]

means that for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that $0 < |t - c| < \delta$, provided

\[ 0 < |t - c| < \delta \Rightarrow |\vec{F}(t) - \vec{L}| < \varepsilon. \]

Let $\vec{F}(t) = f(t)\hat{i} + g(t)\hat{j}$. Then $\vec{F}$ has limit at $c$ if $f$ and $g$ have limits at $c$.

\[ \lim_{t \to c} \vec{F}(t) = [\lim_{t \to c} f(t)]\hat{i} + [\lim_{t \to c} g(t)]\hat{j} \]

**Continuity**

$\vec{F}(t)$ is continuous if $\lim_{t \to c} \vec{F}(t) = \vec{F}(c)$.

**Derivative**

$\vec{F}'(t) = \lim_{h \to 0} \frac{\vec{F}(t + h) - \vec{F}(t)}{h}$
11.5 (continued)

\[ \frac{d}{dt} F(t) = \lim_{h \to 0} \frac{[f(t+h)\hat{t} + g(t+h)\hat{t}]}{h} - [f(t)\hat{t} + g(t)\hat{t}] \]

\[ = \lim_{h \to 0} \left[ \frac{f(t+h) - f(t)}{h} \right] \hat{t} + \lim_{h \to 0} \left[ \frac{g(t+h) - g(t)}{h} \right] \hat{t} \]

\[ \hat{F}(t) = f'(t)\hat{t} + g'(t)\hat{t} \]

**Differentiation Formulas**

1. \( D_x[F(x) + G(x)] = F'(x) + G'(x) \)
2. \( D_x[cf(x)] = cF'(x) \)
3. \( D_x[h(x)F(x)] = h(x)F'(x) + h'(x)F(x) \)
4. \( D_x[F(x)G(x)] = F(x)G'(x) + G(x)F'(x) \) (Product Rule)
5. \( D_x[F(h(x))] = F'(h(x))h'(x) \) (Chain Rule)

\[ \int \hat{F}(t) \, dt = \left[ \int f(t) \, dt \right] \hat{t} + \left[ \int g(t) \, dt \right] \hat{t} \]

**Example 1**

\[ \lim_{t \to \infty} \left[ \frac{t \sin t}{t^2} \right] \hat{t} - \left[ \frac{t^3}{t^2 - 3t} \right] \hat{t} \]

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Ex 2. Find $\ddot{r}'(x) + \dddot{r}''(x)$ for

$\ddot{r}(x) = (e^x + e^{-x})e + 2x^2$.

Ex 3. $\ddot{f}(y) = \tan^2 y e^{\cos(\tan^2 y)}$

Find $\ddot{f}'(y)$. 
\( \mathbf{r}(t) \) is the position vector at any time \( t \) along a curve given by

\[
\begin{align*}
    x &= x(t) \\
    y &= y(t)
\end{align*}
\]

\[
\Rightarrow \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}
\]

**Curvilinear motion** \( \Rightarrow \) the motion associated with tracing the path of the moving \( p \) along the curve.

\[
\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}
\]

(velocity)

\[
\mathbf{a}(t) = \mathbf{v}'(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}
\]

(acceleration)

* Velocity vector is along tangent to curve
* Acceleration vector points to concave side of curve

Ex. 4 (a)  
Given \( \mathbf{r}(t) = 4 \sin t \mathbf{i} + 8 \cos t \mathbf{j} \), find

\( \mathbf{v}(t) \) and \( \mathbf{a}(t) \).
11.5 (continued)

(b) (Ex 4 contd) Find speed when $t = \frac{1}{2}$. 

(c) Sketch a portion of the graph of $\vec{v}(t)$ containing the position $P$ of the particle at $t = \frac{1}{2}$. (Draw $\vec{v}$ and $\vec{a}$ at $P$ as well.)
Ex 5 Suppose that point P moves around a circle w/ center (0,0) & radius r at a constant angular speed of \( \omega \) radians/sec. If its initial position is (0,r), find its acceleration.

We know

\[
\vec{r}(t) = r \cos \omega t \hat{i} + r \sin \omega t \hat{j} \\
\vec{v}(t) = r \sin \omega t \hat{i} + r \cos \omega t \hat{j}
\]

for circular motion.
A 3d curve can be given parametrically by 
\[ x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I, \]
where 
f, g, and h are all continuous on I.

We could specify the curve as the curve traced 
out by the position vector 
\[ \vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}. \]

---

**Lines**

Given a point \( P_0 \) on a line and 
a fixed vector \( \vec{v} = a\hat{i} + b\hat{j} + c\hat{k} \), 
the line is parallel to \( \vec{v} \).

Then 
\[ \vec{r} = \vec{r}_0 + \vec{v}t. \]

And if \( \vec{r} = \langle x, y, z \rangle \) 
\[ \Rightarrow \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t, \quad t \in \mathbb{R}. \]

\[ x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct. \]

**Parametric Eqns of a line in 3d**

\( a, b, c \) are called direction 
\hspace{1pt} \text{numbers} \hspace{1pt}
for the line

any constant multiple of \( a, b, c \) are also 
\hspace{1pt} \text{ direction numbers) \hspace{1pt} \text{numbers} \)
Ex 1: Find parametric eqns of line through 
\((2, 1, 5) + (7, -2, 3)\)

Symmetric Eqns for a line

\[
\begin{align*}
X &= x_0 + at \\
y &= y_0 + bt \\
z &= z_0 + ct
\end{align*}
\]

(assume \(a \neq 0\), \(b \neq 0\))

\[
\begin{align*}
t &= \frac{x - x_0}{a} \\
t &= \frac{y - y_0}{b} \\
t &= \frac{z - z_0}{c}
\end{align*}
\]

This is basically the line of intersection between the two planes given by

\[
\begin{align*}
\frac{x - x_0}{a} &= \frac{y - y_0}{b} \\
\frac{y - y_0}{b} &= \frac{z - z_0}{c}
\end{align*}
\]
Ex 2. Write the symmetric equs for the line through \((-2,2,-2)\) parallel to \((7,-6,37)\).

Ex 3. Find the symmetric equs of the line thru \((5,7,-2)\) \perp to both \((3,1,-37)\) \& \((5,4,-17)\).
Ex 4. Find the symmetric eqns of the line of intersection between the planes 
\[ x + y - 3 = 2 \quad \text{and} \quad 3x - 2y + 3 = 3. \]

**Tangent line to a Curve**

If \( \vec{r} = f(t) \hat{i} + g(t) \hat{j} + h(t) \hat{k} \) is position vector along a curve in 3d, then 
\[ \vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \]

\[ = f'(t) \hat{i} + g'(t) \hat{j} + h'(t) \hat{k} \]

and is in the direction of the tangent line to the 3d curve.
Ex 5  Find the parametric eqns of tangent line
to curve \( x = 2t^2, \ y = 4t, \ z = t^3 \) at \( t = 1 \).
The graph of a 3-variable equation is a surface in 3d. One technique for graphing them is to graph cross-sections (intersections of the surface with well-chosen planes) and/or traces (intersects of the surface with the coordinate planes).

Ex 1 Sketch a graph of \( y^2 + z^2 = 15 \).

If \( x = 0 \), \( y^2 + z^2 = 15 \) is a circle. In fact, regardless of what \( x \) is, \( y^2 + z^2 = 15 \) will be a circle. So, all cross sections are circles centered about \((x, 0, 0)\).

\( \Rightarrow y^2 + z^2 = 15 \) graphs into a right circular cylinder.

In computer graphics, it's common to show many cross sections to display the shape of a surface.
A new defn of cylinder =) the set of all pts on lines \( \parallel \) to \( l \) that intersect \( C \), where \( C \) is a plane curve + \( l \) is a line intersecting \( C \) but not in the plane of \( C \).

**Quadric Surfaces**

A 3d surface whose eqn is of the second degree. The general eqn is

\[
Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fyz + Gx + Hy + Iz + J = 0
\]

But, w/ rotation + translation, these possibilities can be reduced to 2 distinct types

1. \( Ax^2 + By^2 + Cz^2 + J = 0 \)

2. \( Ax^2 + By^2 + Iz^2 = 0 \)
### Quadric Surfaces

#### Ellipsoid:

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy-plane</td>
<td>Ellipse</td>
</tr>
<tr>
<td>xz-plane</td>
<td>Ellipse</td>
</tr>
<tr>
<td>yz-plane</td>
<td>Ellipse</td>
</tr>
<tr>
<td>Parallel to xy-plane</td>
<td>Ellipse, point, or empty set</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Ellipse, point, or empty set</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Ellipse, point, or empty set</td>
</tr>
</tbody>
</table>

![Figure 7](image7.png)

#### Hyperboloid of One Sheet:

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1
\]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy-plane</td>
<td>Ellipse</td>
</tr>
<tr>
<td>xz-plane</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>yz-plane</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>Parallel to xy-plane</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Hyperbola</td>
</tr>
</tbody>
</table>

![Figure 8](image8.png)

#### Hyperboloid of Two Sheets:

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1
\]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy-plane</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>xz-plane</td>
<td>Ellipse, point, or empty set</td>
</tr>
<tr>
<td>yz-plane</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>Parallel to xy-plane</td>
<td>Empty set</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Ellipse, point, or empty set</td>
</tr>
</tbody>
</table>

![Figure 9](image9.png)

#### Elliptic Paraboloid:

\[
z = \frac{x^2}{a^2} + \frac{y^2}{b^2}
\]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy-plane</td>
<td>Point</td>
</tr>
<tr>
<td>xz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>yz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>Parallel to xy-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Parabola</td>
</tr>
</tbody>
</table>

![Figure 10](image10.png)
HYPERBOLIC PARABOLOID: \[ z = \frac{y^2}{b^2} - \frac{x^2}{a^2} \]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy-plane</td>
<td>Intersecting straight lines</td>
</tr>
<tr>
<td>xz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>yz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>Parallel to xy-plane</td>
<td>Hyperbola or intersecting straight lines</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Parabola</td>
</tr>
</tbody>
</table>

ELLIPTIc CONE: \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \]

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy-plane</td>
<td>Point</td>
</tr>
<tr>
<td>xz-plane</td>
<td>Intersecting straight lines</td>
</tr>
<tr>
<td>yz-plane</td>
<td>Intersecting straight lines</td>
</tr>
<tr>
<td>Parallel to xy-plane</td>
<td>Ellipse or point</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Hyperbola or intersecting straight lines</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Hyperbola or intersecting straight lines</td>
</tr>
</tbody>
</table>

Figure 11

Figure 12

Ex 2 Name these graphs

(a) \[ 9x^2 + y^2 - 16z^2 = -25 \]

(b) \[ 9x^2 + y^2 - 16z^2 = 25 \]

(c) \[ x^2 + 4y^2 - 100z = 0 \]

(d) \[ x^2 - y^2 = 0 \]

(e) \[ x^2 - y^2 = 25 \]
11.9 Cylindrical + Spherical Coordinates

**Cartesian**

\[ P(x, y, z) \]

**Cylindrical**

\[ P(r, \theta, z) \]

\[ r \geq 0, \theta \in [0, 2\pi) \]

**Spherical**

\[ P(\rho, \theta, \phi) \]

\[ \rho \geq 0, \theta \in [0, 2\pi), \phi \in [0, \pi] \]

Same pt \( P \) can be described 3 different ways!

**Cylindrical Coords**

From \( \text{Cartesian} \) to \( \text{Cylindrical} \)

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

\[
\begin{align*}
r &= \sqrt{x^2 + y^2} \\
\tan \theta &= y/x \\
z &= z
\end{align*}
\]

**Spherical Coords**

From \( \text{Cartesian} \) to \( \text{Spherical} \)

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi
\end{align*}
\]

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2 + z^2} \\
\tan \theta &= y/x \\
\cos \phi &= \frac{z}{\sqrt{x^2 + y^2 + z^2}}
\end{align*}
\]

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Examples

Cylinders (in cylindrical coords)
- \( r = 1 \)
- \( r = 4 \)

Planes (in cylindrical coords)
- \( \theta = 0, \theta = \frac{\pi}{4}, \theta = \frac{3\pi}{2} \)

Sphere (in spherical coords)
- \( \rho = 4 \)

Ex 1

Change cylindrical coords to Cartesian

(a) \((3, \frac{\pi}{3}, -4)\)

(b) \((2, 2, 3)\)
Ex 2

(a) change from spherical to Cartesian

(8, π/4, π/6)

(b) change from Cartesian to spherical

(2\sqrt{3}, 6, -4)
11.9 (continued)

Ex 3 Change from cylindrical to spherical

\((1, \pi/2, 1)\)

We can figure out relationship from cylindrical to spherical and vice versa.

\[
\begin{align*}
\text{spherical to cylindrical:} & \quad \rho = \rho \sin \phi \\
& \quad \theta = \theta \\
& \quad z = \rho \cos \phi \\
\text{cylindrical to spherical:} & \quad \rho = \sqrt{r^2 + z^2} \\
& \quad \theta = \theta \\
& \quad \cos \phi = \frac{z}{\sqrt{r^2 + z^2}}
\end{align*}
\]
Ex 4. Make required change in given eqn.

(a) \( x^2 - y^2 = 25 \) to cylindrical coords

(b) \( x^2 + y^2 - z^2 = 1 \) to spherical coords

(c) \( \rho = 2 \cos \phi \) to cylindrical coords
(d) \[ x+y+z=1 \] to spherical coords

(e) \[ r=2\sin\theta \] to Cartesian coords

(f) \[ \rho\sin\phi=1 \] to Cartesian coords