9.5 Alternating Series, Absolute Convergence, and Conditional Convergence

Alternating series $\Rightarrow$ every other term has opposite signs

(assume $a_i > 0$) e.g. $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$.

AST (Alternating Series Test)

Let $a_1 - a_2 + a_3 - a_4 + \dots$ be an alternating series w/ $a_n > a_{n+1} > 0$. If $\lim_{n\to\infty} a_n = 0$, then series converges.

And error made by estimating sum to be S_n is less than or equal to a_{n+1}, i.e.

$$E = |S - S_n| \le a_{n+1}.$$

pf

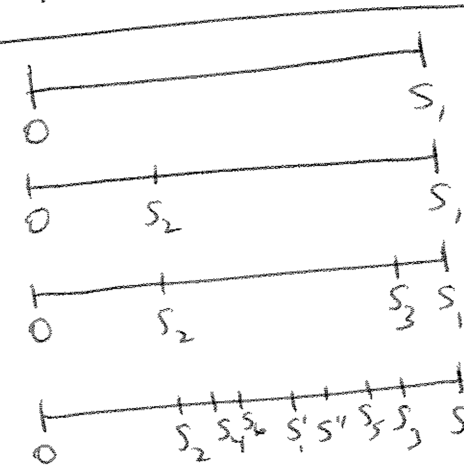
$S_1 = a_1$

$S_2 = a_1 - a_2 = S_1 - a_2$

$S_3 = a_1 - a_2 + a_3 = S_2 + a_3$

$S_4 = a_1 - a_2 + a_3 - a_4 = S_3 - a_4$

$\vdots$



$S_2, S_4, S_6, \dots$ are increasing & bounded above (since $a_n > a_{n+1}$). Call $\lim_{n\to\infty} S_{2n} = S'$.

Also, $S_1, S_3, S_5, \dots$ are decreasing & bndd below, call it $\lim_{n\to\infty} S_{2n+1} = S''$.

9.5 (cont)

You can see that S' & S'' are in between S_n and S_{n+1} $\forall n$.

$$\Leftrightarrow \quad |S''-S'| \leq |S_{n+1}-S_n| = a_{n+1}.$$

But we know $\lim_{n\to\infty} a_{n+1} = 0.$

$$\Rightarrow \lim_{n\to\infty} |S''-S'| = 0 \Rightarrow S''=S', \text{ i.e. both}$$

the even #d sums & odd #d sums converge to same limit, and $|S-S_n| \leq |S_{n+1}-S_n| = a_{n+1}$ //

Ex1 (Alternating Harmonic Series)

converge or diverge? $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

9.5 (cont)

Ex 2 Diverge or converge? $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$

What is error estimate made by approximating S by S_6?

$\boxed{\begin{array}{l}\text{Absolute Convergence Test} \\ \text{If } \sum |u_n| \text{ converges, then } \sum u_n \text{ converges.}\end{array}}$

9.5 (cont)

Ex 3 Does $2 + \frac{2}{2^3} + \frac{2}{3^3} - \frac{2}{4^3} + \frac{2}{5^3} + \frac{2}{6^3} + \frac{2}{7^3} - \frac{2}{8^3} + \cdots$

converge or diverge?

Absolute Ratio Test

Let $\sum u_n$ be a series of nonzero terms and suppose $\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \rho$.

If
1. $\rho < 1$, series converges absolutely.
2. $\rho > 1$, series diverges.
3. $\rho = 1$, test is inconclusive

9.5 (cont)

Ex 4 Show $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{e^n}$ converges absolutely.

Conditional Convergence

$\sum u_n$ is conditionally convergent if $\sum u_n$ converges but $\sum |u_n|$ diverges.

Ex 5 Classify as absolutely convergent, conditionally convergent or divergent. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1} + \sqrt{n}}$

(☆ more space on next page)

Ex5 (cont)

Rearrangement Thm

Terms of an absolutely convergent series can be rearranged w/o affecting either the convergence or the sum of the series.

★ Notice that it's **not** true for conditionally convergent series.

9.6 Power Series

Now, we'll consider a series of functions instead of constants.

Power Series in x

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$ (we can think of a_0 as $a_0 x^0$)

Ex 1 When does this power series converge, i.e. for what x values? $\sum_{n=0}^{\infty} a x^n$ $\quad a \in \mathbb{R}$, $a \neq 0$.

9.6 (cont) $\quad$ convergence set $\Rightarrow$ set of x-values where power series converges

Thm

The convergence set for a power series $\sum a_n x^n$ is always an interval of one of these 3 types

① The single pt at $x=0$.

② an interval $(-R,R)$, $[-R,R]$, $[-R,R)$ or $(-R,R]$ for some $R \in \mathbb{R}$

③ $(-\infty, \infty)$

The radius of convergence is 0, R or ∞, respectively.

Ex 2 Find convergence set for

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

9.6 (cont)

Ex 3 Find convergence set for

$$1 + x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \frac{x^4}{\sqrt{4}} + \cdots$$

9.6 (cont)

Power Series in $(x-a)$

$$\sum a_n (x-a)^n = a_0 + a_1 (x-a) + a_2 (x-a)^2 + \dots$$

convergence set: ① single pt at $x=a$.

② interval $(a-R, a+R)$ (and maybe endpts)

③ $(-\infty, \infty)$

Ex 4 Find convergence set for

$$\frac{x-3}{2} + \frac{(x-3)^2}{2^2} + \frac{(x-3)^3}{2^3} + \dots$$

9.7 Operations on Power Series

Think of a power series as a polynomial w/ ∞ly many terms.

Thm A

Let $S(x) = \sum_{n=0}^{\infty} a_n x^n$ on interval I.

If x is interior to I, then

① $S'(x) = \sum_{n=0}^{\infty} D_x(a_n x^n) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

and ② $\int_0^x S(t)\,dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

i.e. we can differentiate + integrate a power series and radius of convergence is the same for $S(x)$, $S'(x)$ and $\int_0^x S(t)\,dt$!!!

Ex 1 We know $1+x+x^2+x^3+\dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ (geometric series)

$x \in (-1,1)$

$\Rightarrow \int_0^x \frac{1}{1-t}\,dt = \sum_{n=0}^{\infty} \int_0^x t^n\,dt = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$

and $\int_0^x \frac{1}{1-t}\,dt = \int_1^{1-x} -\frac{1}{u}\,du = -\ln|u| \Big]_1^{1-x}$

$= -(\ln|1-x| - \ln 1)$

$= -\ln|1-x|$

$u = 1-t$

$du = -dt$

$-du = dt$

$t=0,\ u=1$

$t=x,\ u=1-x$

$|1-x| = 1-x$ (since $-1<x<1 \Rightarrow 1-x>0$)

$\Rightarrow \boxed{-\ln(1-x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}}$

9.7 (cont)

$$\Rightarrow -\ln(1-(-x)) = -\ln(1+x)$$
$$= \sum_{n=0}^{\infty} \frac{1}{n+1}(-x)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} x^{n+1}$$

$$\Rightarrow \ln(1+x) = -(-\ln(1+x)) = \sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{n+1} x^{n+1}$$

i.e. $$\boxed{\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}}$$

Ex 2 Show $S'(x) = S(x)$ for

$$S(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots .$$

(You first need to convince yourself that this converges.)

Then solve $S(x) = S'(x)$. (notice $S(0) = 1$)

9.7 (cont)

$$\Rightarrow \boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.}$$

We also can derive
$$\boxed{\begin{gathered}\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \\ \forall\, x \in (-1,1)\end{gathered}}$$

Ex 3 Find power series for $f(x) = \frac{x}{(1+x)^2}$

9.7 (cont)

Thm B

If $f(x) = \sum a_n x^n$ & $g(x) = \sum b_n x^n$ w/ both series converging for $|x| < r$, we can perform arithmetic operations and the resulting series will converge for $|x| < r$. (If $b_0 \neq 0$, result holds for division, but we can guarantee its validity only for $|x|$ sufficiently small.)

Ex 4 Find power series for $f(x) = \sinh(x)$

9.7 (cont)

Ex 5 Find power series for $f(x) = \dfrac{\arctan(x)}{1+x^2+x^4}$

9.7 (cont)

Ex 6 Find these sums.

(a) $1+x^2+x^4+x^6+x^8+\dots = ?$

(b) $\cos x+\cos^2 x+\cos^3 x+\cos^4 x+\dots = ?$

9.8 Taylor and Maclaurin Series

If we can represent some function $f(x)$ as a power series in $(x-a)$, then

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

$$\Rightarrow f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$$

$$f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + 20c_5(x-a)^3 + \dots$$

$$f'''(x) = 3\cdot 2\, c_3 + 4\cdot 3\cdot 2\, c_4(x-a) + 5\cdot 4\cdot 3\, c_5(x-a)^2 + \dots$$

$$\vdots$$

Let $x=a \Rightarrow f'(a) = c_1 \quad f''(a) = 2c_2, \quad f'''(a) = 6c_3, \dots$

$$\Rightarrow c_1 = f'(a), \quad c_2 = \tfrac{1}{2} f''(a), \quad c_3 = \tfrac{1}{6} f'''(a), \dots$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$\Rightarrow$ each c_n is unique and dependent on the function

Uniqueness Thm

Suppose $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$ $\forall x$ in some interval around a.

Then $c_n = \dfrac{f^{(n)}(a)}{n!}$

$\star$ called Taylor series
(if $a=0$, it's called Maclaurin series)

9.8 (cont)

We still have questions about the existence of a power series representation $\forall$ fns.

Taylor's Formula w/ Remainder

Let $f(x)$ be a function $\ni$ $f^{(n+1)}(x)$ exists $\forall x \in I$ (I = open interval containing a).

Then $\forall x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x).$$

where $R_n(x)$ is the remainder (or error)

& $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ where c is some value between x and a.

Taylor's Thm

Let f be function w/ all derivatives in $(a-r, a+r)$.

The Taylor Series $f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

represents $f(x)$ on $(a-r, a+r)$ $\Leftrightarrow$ $\lim\limits_{n\to\infty} R_n(x) = 0$ where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

$c \in (a-r, a+r)$.

9.8 (cont)

Ex 1 Find Maclaurin series for $f(x) = \cos x$ + prove it represents $\cos x$ $\forall x$.

9.8 (cont)

Ex 2 Find Maclaurin series for $f(x) = \sin x$.

Binomial Series

We know $(1+x)^p = \underbrace{1 + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \dots + \binom{p}{p}x^p}_{\text{Binomial Formula}}$

(can get $\binom{p}{n}$ from Pascal's Δ)

By defn, $\binom{p}{k} = {}_pC_k = \dfrac{p!}{k!(p-k)!} = \dfrac{p(p-1)(p-2)\dots(p-(k-1))\cancel{(p-k)!}}{k!\,\cancel{(p-k)!}}$

$\Rightarrow \binom{p}{k} = \dfrac{p(p-1)(p-2)\dots(p-k+1)}{k!}$ + this is true $\forall\, p \in \mathbb{R}$ (k still must be in $\mathbb{Z}^+$)

$$\boxed{\begin{array}{l}\forall\, p \in \mathbb{R} \text{ + } |x|<1 \\ (1+x)^p = 1 + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \dots\end{array}}$$

9.8 (cont)

(Note: If $p \in \mathbb{Z}^+$, then $\binom{p}{k} = 0 \ \forall\ k > p$ which means our infinite series collapses to a finite sum + we get binomial formula as we expect.)

Ex 3 Write the Maclaurin series for
$f(x) = (1-x^2)^{2/3}$ (through first 5 terms)

9.8 (cont)

Ex 4 Find Taylor Series for $f(x) = \sin x$ in $(x - \pi/4)$.

9.8 (cont)

Ex 5 Use what we already know to write Maclaurin series for $f(x) = \frac{1}{1-\sin x}$ (up to x^5 term)

9.9 The Taylor Approximation to a Function

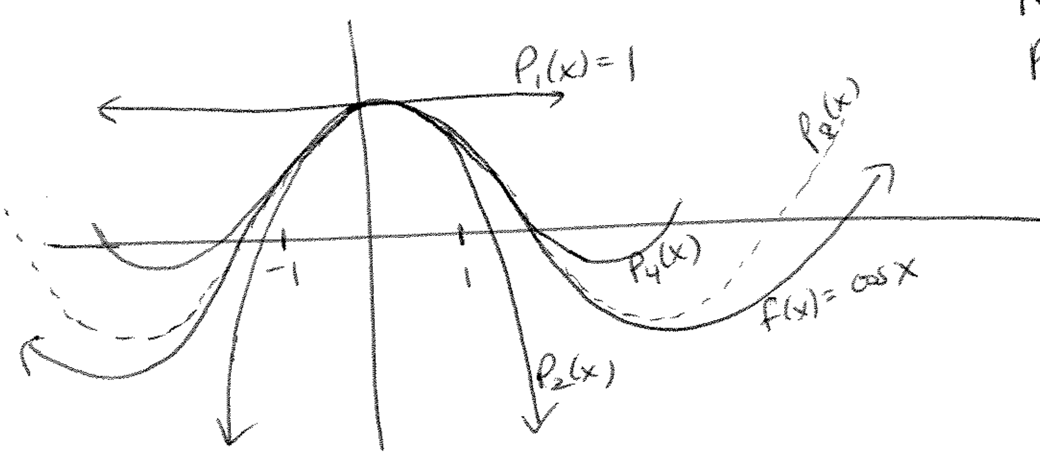
Many math problems that occur in applications cannot be solved exactly, like $\int_0^b \sin(x^2)\,dx$. We need to approximate them!

Taylor Polynomial of order n (based at a)

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(order $\Rightarrow$ the last derivative that we take.
order $\neq$ degree of polynomial necessarily.)

☆ We can get the Maclaurin polynomial by using $a=0$.



Notice $P_8(x)$ reasonable fit for $\cos x$ on $(-1, 1)$ (and a little beyond)

9.9 (cont)

Ex 1 For $f(x) = e^{-3x}$, find the Maclaurin polynomial of order 4 + approximate $f(0.12)$.

Lagrange Error for Taylor Polynomials

We know $f(x) = P_n(x) + R_n(x)$

Taylor polynomial of order n

remainder/error

and $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ $\quad c \in (x, a)$

9.9 (cont)

Ex 2 Find error in estimating $f(0.12)$ in last example. $R_4(x) = \frac{f^{(5)}(c)(x^5)}{5!}$

$f(x) = e^{-3x}$

☆ This gives us error bound for the method. But there are also rounding errors along the way from computations.

$\Rightarrow$ There's a balance for errors since more terms reduces errors in method but increases error in calculations.

9.9 (cont)

① Look at $s - a_1 - a_2 - a_3 - a_4 - \cdots - a_n$ where $a_i = 0.001$ and $s = 1{,}000{,}000$. If we start w/ $s - a_1$ then $(s-a_1)-a_2$, then $((s-a_1)-a_2)-a_3$, etc. we lose to error problems quickly & stay around 1 million.

② But if we add $a_1 + a_2 + a_3 + \cdots + a_{10{,}000} = 10$
& do $s - (a_1 + a_2 + \cdots + a_{10{,}000}) = 1{,}000{,}000 - 10 = 999{,}990$

$\Rightarrow$ ② is better way to avoid errors in this case...

To find a good bound for $R_n(x)$, we can use the triangle inequality $|a \pm b| \le |a| + |b|$.

Ex 3 Find a good bound for max value of $\left|\frac{4c}{c+4}\right|$ given $c \in [0,1]$

9.9 (cont)

Ex 4 Find a good bound for max value of $\left|\frac{c^2-c}{\cos c}\right|$ given $c \in [0, \pi/4]$

Ex 5 $n = ?$ $\ni$ Maclaurin polynomial for $f(x) = e^x$ has $f(1)$ approximated to five decimal places, i.e. $|R_n(1)| \le 0.000005$.