4.6 Numerical Integration

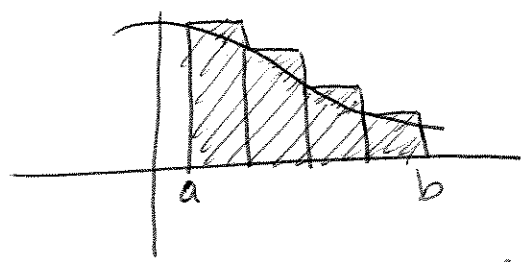
If $f(x)$ is continuous, we are guaranteed $\int_a^b f(x)\,dx$ exists. But sometimes we cannot evaluate it. For these cases, we use numerical methods to approximate definite integral.

Methods

① **Left Riemann Sum**: area of n^{th} rectangle $= f(x_{n-1})\Delta x_n$

$x_{i-1} = a + (i-1)\Delta x$ $\quad \Delta x = \frac{b-a}{n}$ $\quad \int_a^b f(x)\,dx \simeq \boxed{\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+(i-1)\left(\frac{b-a}{n}\right)\right)}$

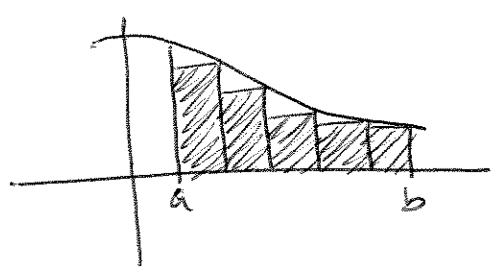
$E_n = \frac{(b-a)^2}{2n} f'(c)$ for some $c \in [a,b]$.



② **Right Riemann Sum**: area of n^{th} rectangle $= f(x_n)\Delta x_n$

$\Delta x = \frac{b-a}{n}$ $\quad x_i = a + i\Delta x$ $\quad \int_a^b f(x)\,dx \simeq \boxed{\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+i\left(\frac{b-a}{n}\right)\right)}$

$E_n = \frac{-(b-a)^2}{2n} f'(c)$ for some $c \in [a,b]$



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③ **Midpoint Riemann Sum**: area of i^{th} rectangle $=$

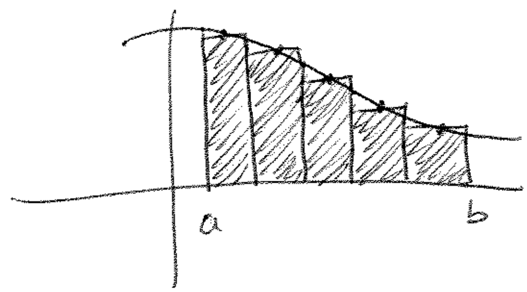
$$f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x_i$$

$$\Delta x = \frac{b-a}{n} \qquad \begin{matrix} x_i = a + i\Delta x \\ x_{i-1} = a+(i-1)\Delta x \end{matrix} \Bigg\rangle \ \frac{x_{i-1}+x_i}{2} = a + i\Delta x - \tfrac{1}{2}\Delta x$$

$$\Rightarrow \int_a^b f(x)\,dx \simeq \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\Delta x - \tfrac{1}{2}\Delta x\right)$$

$$= \boxed{\frac{b-a}{n}\sum_{i=1}^{n} f\left(a + \left(i-\tfrac{1}{2}\right)\left(\frac{b-a}{n}\right)\right)}$$

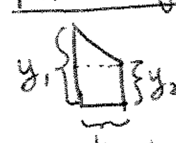
$$E_n = \frac{(b-a)^3}{24n^2} f''(c) \quad \text{for some } c \text{ in } [a,b].$$



④ **Trapezoidal Rule**:

Σ area of i^{th} trapezoid $= \frac{1}{2}(f(x_i)+f(x_{i-1}))\Delta x_i$

Area of trapezoid



$A = \frac{1}{2}(y_1 - y_2)h + hy_2$

$A = \frac{1}{2}(y_1+y_2)h$

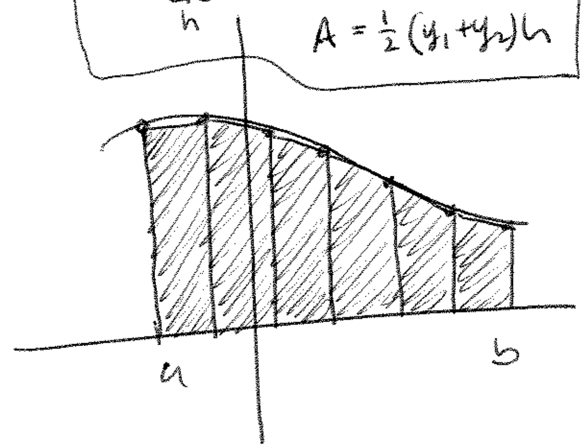
$$\Delta x_i = \frac{b-a}{n} \qquad x_i = a + i\Delta x \qquad x_{i-1} = a + (i-1)\Delta x_i$$

$$\int_a^b f(x)\,dx \simeq \frac{1}{2}\left(\frac{b-a}{n}\right)\sum_{i=1}^{n}\left[f(x_{i-1}) + f(x_i)\right]$$

$$= \frac{1}{2}\left(\frac{b-a}{n}\right)\Big[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + f(x_3) + \dots + f(x_{n-1}) + f(x_{n-1}) + f(x_n)\Big]$$

$$= \frac{1}{2}\left(\frac{b-a}{n}\right)\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\Big]$$

$$= \boxed{\left(\frac{b-a}{n}\right)\left[\frac{f(a)}{2} + \sum_{i=1}^{n-1} f(x_i) + \frac{f(b)}{2}\right]}$$



4.6 (cont)

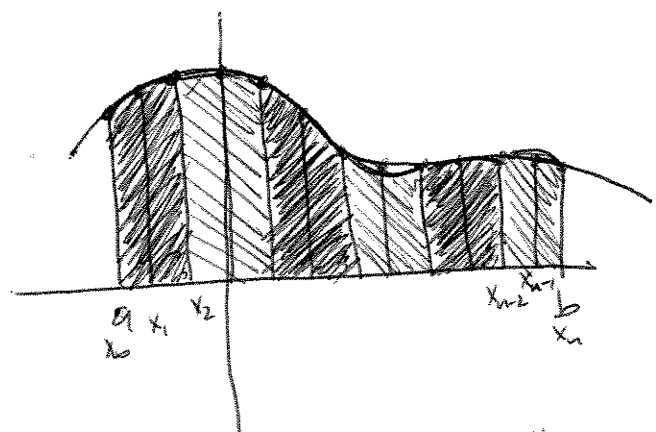
(④ Trapezoidal Rule cont)

$$E_n = -\frac{(b-a)^3}{12n^2} f''(c) \quad \text{for some } c \in [a,b]$$

⑤ Parabolic Rule (aka Simpson's Rule) (★ n must be even)

$\Delta x = \frac{b-a}{n}$ $\quad x_i = a + i\Delta x$ $\qquad x_0 = a,\ x_n = b$

$\Sigma \square$



area of one parabolic piece =

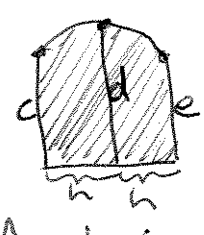
$$\frac{\Delta x}{3}\left(f(x_i) + 4f(x_{i+1}) + f(x_{i+2})\right)$$

For every two "widths", we connect top 3 pts w/ parabola.

$$\int_a^b f(x)\,dx \simeq \left(\frac{b-a}{n}\right)\left(\frac{1}{3}\right)\Big[\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \left(f(x_2) + 4f(x_3) + f(x_4)\right) + \left(f(x_4) + 4f(x_5) + f(x_6)\right) + \dots + \left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)\Big]$$

$$= \boxed{\frac{b-a}{3n}\left[f(a) + 4\sum_{i=1}^{n/2} f(a + (2i-1)\Delta x) + 2\sum_{i=1}^{n/2-1} f(a + 2i\Delta x) + f(b)\right]}$$

Area of parabolic piece



$$A = \frac{h}{3}(c + 4d + e)$$

★ see proof on pg Ⓕ

$$E_n = -\frac{(b-a)^5}{180n^4} f^{(4)}(c) \quad \text{for some } c \in [a,b]$$

4.6 (cont)

Ex 1 Use methods ②, ④ + ⑤ to approximate

$\int_1^3 \frac{1}{x^3}\,dx$, let $n=8$.

Actual answer

$\int_1^3 x^{-3}\,dx$

$= \frac{x^{-2}}{-2}\Big|_1^3$

$= \frac{-1}{2x^2}\Big|_1^3$

$= \frac{-1}{2(9)} - \frac{-1}{2}$

$= \frac{-1}{18} + \frac{1}{2}$

$= \frac{8}{18} = \frac{4}{9} = 0.\overline{4}$

② Right Riemann Sum:

④ Trapezoidal method:

(more space on next page)

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4.6 (cont)

Ex 1 (cont)

(5) Parabolic method:

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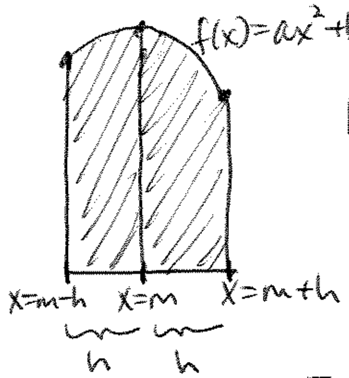
(E)

4.6 (cont) Area of Parabolic piece Proof



Area under curve $= \int_{m-h}^{m+h} (ax^2+bx+c)\,dx$

$$= \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right)\Big|_{m-h}^{m+h}$$

$$= \left(\frac{a}{3}(m+h)^3 + \frac{b}{2}(m+h)^2 + c(m+h)\right) - \left(\frac{a}{3}(m-h)^3 + \frac{b}{2}(m-h)^2 + c(m-h)\right)$$

$$= \frac{a}{3}(m^3+3m^2h+3mh^2+h^3) + \frac{b}{2}(m^2+2mh+h^2) + cm + ch$$
$$- \frac{a}{3}(m^3-3m^2h+3mh^2-h^3) - \frac{b}{2}(m^2-2mh+h^2) - cm + ch$$

$$= \cancel{\frac{a}{3}m^3} + am^2h + \cancel{amh^2} + \frac{a}{3}h^3 + \cancel{\frac{b}{2}m^2} + bmh + \cancel{\frac{b}{2}h^2} + \cancel{cm} + ch$$
$$- \cancel{\frac{a}{3}m^3} + am^2h - \cancel{amh^2} + \frac{a}{3}h^3 - \cancel{\frac{b}{2}m^2} + bmh - \cancel{\frac{b}{2}h^2} - \cancel{cm} + ch$$

$$= 2\left(am^2h + \frac{a}{3}h^3 + bmh + ch\right) = 2h\left(am^2 + \frac{1}{3}ah^2 + bm + c\right)$$

Our claim is that area $= \frac{h}{3}\left[f(m-h) + 4f(m) + f(m+h)\right]$

Let's check $\frac{h}{3}\left[f(m-h) + 4f(m) + f(m+h)\right]$

$$= \frac{h}{3}\left\{(a(m-h)^2 + b(m-h) + c) + 4(am^2 + bm + c) + (a(m+h)^2 + b(m+h) + c)\right\}$$

$$= \frac{h}{3}\left[am^2 - \cancel{2amh} + ah^2 + bm - \cancel{bh} + c + 4am^2 + 4bm + 4c + am^2 + \cancel{2amh} + ah^2 + bm + \cancel{bh} + c\right]$$

$$= \frac{h}{3}\left\{6am^2 + 2ah^2 + 6bm + 6c\right\} = h\left[2am^2 + \frac{2}{3}ah^2 + 2bm + 2c\right]$$

$$= 2h\left[am^2 + \frac{1}{3}ah^2 + bm + c\right] \quad //$$

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