6.1 Antiderivatives & Indefinite Integrals

We are now onto “undoing” derivatives. If I know \( \frac{dy}{dx} = 2x \), what is \( y \)?

\[
\begin{align*}
y &= x^2 \
y &= x^2 + 1 \
y &= x^2 - 5
\end{align*}
\]

In general, \( y = x^2 + c \) where \( c \in \mathbb{R} \).

**Theorem**

If \( F \) and \( G \) are differentiable functions on \((a, b)\) and \( F'(x) = G'(x) \) \( \forall x \in (a, b) \), then \( F(x) = G(x) + k \) for some constant \( k \).

i.e. any antiderivative (a.k.a. indefinite integral) returns a family of solutions

**Notation:** If \( F'(x) = f(x) \), then \( \int f(x) \, dx + C \).

\[ \int \]

**Ex 1** \( \int (x^2 + 1) \, dx \) (guess)
6.1 (cont.)

**Indefinite Integrals of Basic Fns**

1. \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \quad C \in \mathbb{R} \)

2. \( \int e^x \, dx = e^x + C \)

3. \( \int \frac{1}{x} \, dx = \ln|x| + C \quad x \neq 0 \)

**Indefinite Integral is a Linear Operator**

1. \( \int k f(x) \, dx = k \int f(x) \, dx \)

2. \( \int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx \)

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**Ex 2**

\[ \int \frac{2}{x^4} \, dx \]

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**Ex 3**

\[ \int \left(7 + \frac{9}{x}\right) \, dx \]
Ex. 4

(a) \( \int (x^2 - 5)(x + 2) \, dx \)

(b) \( \int 3 \sqrt{w^4} \, dw \)

(c) \( \int \frac{1-y^2}{3y} \, dy \)

(d) \( \int e^x - 2x \, dx \)
Ex 5

(a) \( \frac{du}{dt} = \frac{4}{t} + \frac{t}{4} \)

(b) \( \int \frac{(2+y^2)^2}{y} \, dy \)

(c) Find antiderivative of \( \frac{dy}{dx} = \frac{\sqrt{x^3} - 1}{\sqrt{x^2}} \) \( \Rightarrow y(9) = 4 \).
6.2 Integration by Substitution

\[ \int f'[g(x)]g'(x)\,dx = f[g(x)] + C \]

"U-substitution" basically undoes chain rule

Ex. 1 \[ \int x^2 e^{x^3-1} \, dx \]

Ex. 2 \[ \int 5(5x-1)^20 \, dx \]
Ex 3

(a) \( \int \frac{1}{3x-2} \, dx \)

(b) \( \int e^{8x+1} \, dx \)

(c) \( \int \frac{x}{\sqrt{x-5}} \, dx \)
6.2 (cont.)

Example 4
(a) \( \int e^{-x} (1-e^{-x})^4 \, dx \)

(b) \( \int \frac{1}{x \ln x} \, dx \)

(c) \( \int \frac{x^2 - 1}{(x^3 - 3x + 7)^2} \, dx \)
6.4 The Definite Integral

We want to approximate area under a curve. We can overestimate or underestimate with rectangles.

Error in approximating

If \( f(x) > 0 \) and is either increasing or decreasing on \((a, b)\), then the error in estimating area by rectangles is

\[
E = \left| \int_a^b f(x) \, dx - \sum_{i=1}^{n} f(x_i) \Delta x \right| \to 0 \quad \text{as} \quad n \to \infty.
\]

This means we can get to the exact area by over- or under-estimating the area of rectangles and then letting the width of our rectangles be infinitesimally small.

EX Consider \( y = x^2 + 2 \). Find approximate area under curve on \([1, 3]\) by dividing interval into 4 equal sub-intervals.

We will choose right endpoints of intervals.

\[
A = f(1.5) \Delta x + f(2) \Delta x + f(2.5) \Delta x + f(3) \Delta x
\]

\[
\Delta x = \frac{3 - 1}{4} = \frac{2}{4} = \frac{1}{2}
\]
6.4 (cont)

\[ A = (1.5^2 + 2)0.5 + (2^2 + 2)0.5 + (2.5^2 + 2)(0.5) + (3^2 + 2)(0.5) \]

\[ = 4.25(0.5) + 6(0.5) + 8.25(0.5) + 11(0.5) \]

\[ A = 14.75 \]

(This is called a **Riemann sum**.)

**Definite Integral**

Let \( f \) be a continuous function on \([a, b]\). The limit of the Riemann sums for \( f \) on \([a, b]\) is guaranteed to exist if \( f \) is bounded on \([a, b]\). The notation is

\[ \int_a^b f(x) \, dx . \]

A region above the \( x \)-axis is positive, below the \( x \)-axis it's negative.

**Ex.**

**Given**

\[ A = 2 \quad B = 2.5 \quad C = 1.8 \quad D = 4 \]

**Find**

(a) \( \int_b^c f(x) \, dx \)  
(b) \( \int_a^d f(x) \, dx \)  
(c) \( \int_c^e f(x) \, dx \)
Properties of Definite Integrals

1. \( \int_{a}^{a} f(x) \, dx = 0 \)
2. \( \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \)
3. \( \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \)
4. \( \int_{a}^{b} k f(x) \, dx = k \int_{a}^{b} f(x) \, dx \)
5. \( \int_{a}^{b} [f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \)

Ex 2. Evaluate, given \( \int_{1}^{4} x \, dx = 7.5 \), \( \int_{1}^{4} x^2 \, dx = 21 \), and \( \int_{4}^{5} x^2 \, dx = \frac{61}{3} \)

(a) \( \int_{1}^{4} (7x - 2x^2) \, dx \)

(b) \( \int_{1}^{4} x (1-x) \, dx \)

(c) \( \int_{5}^{5} (10 - 4x + 2x^2) \, dx \)
6.5 Fundamental Theorem of Calculus

If \( f \) is continuous on \([a, b]\) and \( F \) is any antiderivative of \( f \), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

Ex 1 \[
\int_{1}^{2} 3x^2 \, dx
\]

Ex 2 \[
\int_{-1}^{2} (x^2 - 4x) \, dx
\]
Ex 3 (a) \[ \int_{0}^{4} 9x^{\frac{1}{2}} \, dx \]

(b) \[ \int_{0}^{1} 32x (x^{2} + 1)^{3} \, dx \]

(c) \[ \int_{0}^{2} xe^{x^{2}} \, dx \]
Ex. 4  (a) $\int_{1}^{2} \frac{x+1}{2x^2 + 4x + 4} \, dx$

(b) $\int_{2}^{8} \frac{1}{x+1} \, dx$
6.5 (cont)

**Defn** Average Value of a continuous function \( f(x) \) on \([a, b]\)

\[
= \frac{1}{b-a} \int_{a}^{b} f(x) \, dx
\]

**Ex 5** Find any value over indicated interval.

(a) \( f(x) = 4x - 3x^2 \) on \([-2, 2]\)

(b) \( g(x) = 2x + 7 \) on \([0, 5]\)