3.1 Limits

Defn of a limit of a function

If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then $\lim_{x \to c} f(x) = L$ ("the limit of $f(x)$ as x approaches c is L").

Properties of limits

$b, c \in \mathbb{R}$ $\quad n \in \mathbb{Z}^+$

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ (if n even, then c positive)

Always true ↑

5. $\lim_{x \to c} [b f(x)] = b \lim_{x \to c} f(x)$
6. $\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
7. $\lim_{x \to c} (f(x) g(x)) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$
8. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ (assuming $\lim_{x \to c} g(x) \neq 0$)
9. $\lim_{x \to c} [f(x)]^n = [\lim_{x \to c} f(x)]^n$
10. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$ (if n even, $\lim_{x \to c} f(x)$ must be positive)

True **only if** $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist

Math1100

3.1 (cont)

Ex 1 Find the limits

(a) $\lim_{x \to 1} (2x^2 - 3x + 1)$

(b) $\lim_{x \to -2} \frac{3x+1}{2-x}$

(c) $\lim_{x \to 5} \frac{\sqrt{x+4} - 2}{5}$

Indeterminate Form

If $\lim_{x \to c} f(x) = 0$ + $\lim_{x \to c} g(x) = 0$,

then $\lim_{x \to c} \frac{f(x)}{g(x)}$ is indeterminate,

(i.e. the $\frac{0}{0}$ case)

(This means there's more to do!)

If $\lim_{x \to c} f(x) = L$, $L \neq 0$,

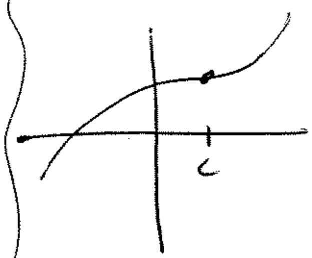
$\lim_{x \to c} g(x) = 0$, then

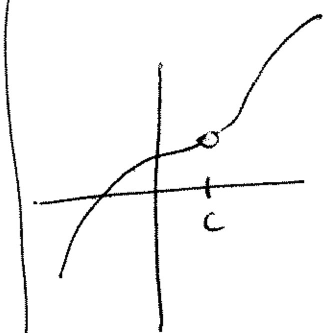
$\lim_{x \to c} \frac{f(x)}{g(x)}$ DNE.

3.1 (cont)

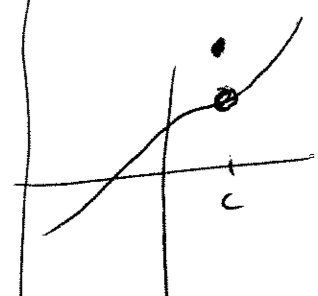
Ex 2 Find the limits.

(a) $\displaystyle\lim_{x\to 3} \frac{x^3-27}{x-3}$

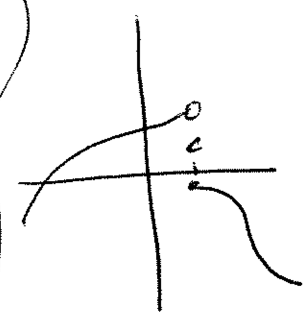




(b) $\displaystyle\lim_{x\to 3} \frac{2x^2-x-15}{x-3}$



(c) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x}$



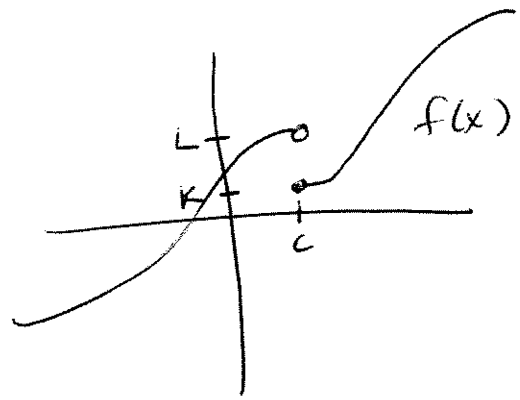
(d) $\displaystyle\lim_{x\to 1} \frac{x+5}{x-1}$

3.1 (cont)

One sided Limits

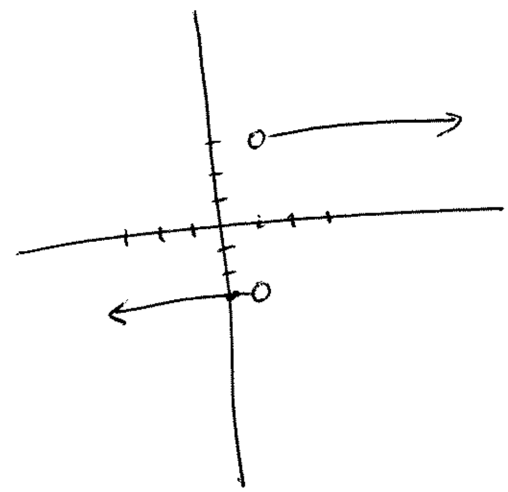
LH Limit $\lim_{x \to c^-} f(x) = L$

RH Limit $\lim_{x \to c^+} f(x) = K$



Ex 3

(a) $\lim_{x \to 1^-} \frac{|3x-3|}{x-1}$



(b) $\lim_{x \to 1^+} \frac{|3x-3|}{x-1}$

3.1 (cont)

$\boxed{\begin{array}{l}\text{If } f \text{ is a function and } c, L \in \mathbb{R}, \text{ then} \\ \lim_{x\to c} f(x) = L \quad \text{iff} \quad \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L.\end{array}}$

Ex 4 For $f(x) = \begin{cases} 2x^2 - 1 & x > 0 \\ 3x - 1 & x < 0 \end{cases}$

find $\lim_{x\to 0} f(x)$.

Ex 5 (a) $\lim_{x\to 3} \frac{5}{x-3}$

(b) $\lim_{x\to 3^-} \frac{5}{x-3}$

(c) $\lim_{x\to 3^+} \frac{5}{x-3}$

3.1 (cont)

Ex 6 $\lim_{x \to 1^-} \frac{2}{x^2-1}$

$\lim_{x \to 1^+} \frac{2}{x^2-1}$

$\lim_{x \to 1} \frac{2}{x^2-1}$

3.2 Continuity (☆ skip pgs 149-151 in book)

Defn of Continuity

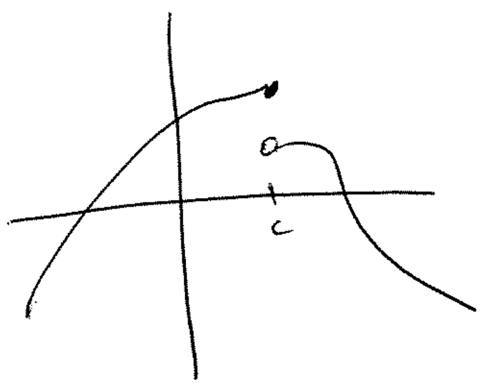
A function $f(x)$ is continuous at $x=c$ if

$$\lim_{x \to c} f(x) = f(c).$$

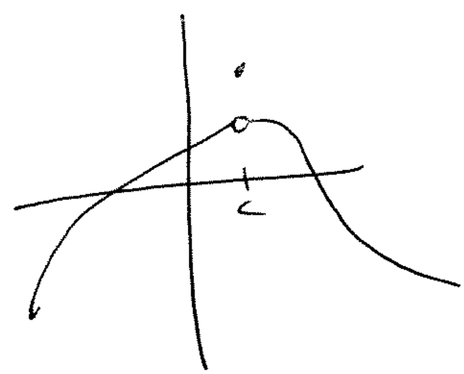
That is, $f(c)$ must exist

$\lim_{x \to c} f(x)$ must exist

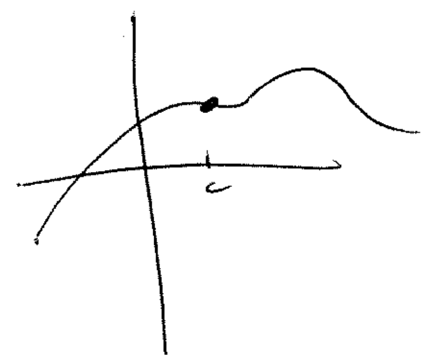
<u>and</u> they have to be equal.



$f(c)$ exists
but $\lim_{x \to c} f(x)$ dne



$f(c)$ exists and
$\lim_{x \to c} f(x)$ exists
but they're not the same



$f(c)$ exists
$\lim_{x \to c} f(x)$ exists
and $\lim_{x \to c} f(x) = f(c)$.

- All polynomials are continuous everywhere.
- All rational functions are continuous everywhere, except where the denominator is zero.

3.2 (cont)

Ex 1 Determine where these functions are continuous.

(a) $f(x) = 3x^3 - 4x^2 + x + 5$

(b) $f(x) = \dfrac{x+4}{x^2 - 6x + 5}$

(c) $f(x) = \dfrac{2}{x^2 - 1}$

(d) $f(x) = \begin{cases} x^2 - 4 & x \le 0 \\ 3x + 1 & x > 0 \end{cases}$

(e) $h(x) = f(g(x))$, $f(x) = \dfrac{1}{x-1}$ $\quad g(x) = x^2 + 5$

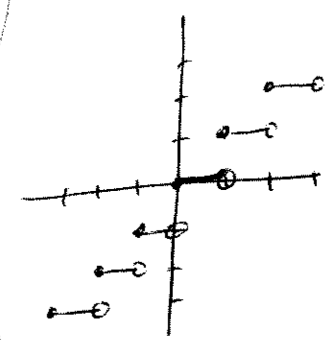
(f) $g(x) = \sqrt{5 - x}$

3.2 (cont)

Ex 2 Describe where f(x) is continuous.

$f(x) = x - [[x]]$

$f(x) = [[x]]$



greatest integer function

Ex 3 (#38) Discuss continuity of $f(x) = \frac{x}{x^2 - 4x + 3}$ on $[0, 4]$. If there are any discontinuities, determine if it's removable.

3.3 Infinite limits & Limits at Infinity

Infinite limits & Vertical Asymptotes

The vertical line $x=a$ is a vertical asymptote for the graph of $y=f(x)$ if

$f(x) \to \pm\infty$ as $x \to a^+$ or $x \to a^-$.

V.A. of Rational Fns

If $f(x) = \dfrac{n(x)}{d(x)}$ is a rational function, $d(c)=0$ & $n(c) \neq 0$, then $x=c$ is a VA of f.

Ex 1 Find vertical asymptotes for

$$f(x) = \frac{x^2+4}{(x-1)(x+3)}$$

3.3 (cont)

Thm 2 If p is a positive $\mathbb{R}$# and $k \in \mathbb{R}$, then

① $\lim\limits_{x \to \pm\infty} \dfrac{k}{x^p} = 0$ ② $\lim\limits_{x \to \pm\infty} k x^p = \pm\infty$

Limits of Rational Fns at ∞ + HA of Rational Fns

(A) If $f(x) = \dfrac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0}$ $a_m \neq 0$, $b_n \neq 0$,

then $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \dfrac{a_m x^m}{b_n x^n}$.

(B) H.A. (aka Horizontal Asymptotes)

3 cases

① If $m < n$, then $\lim\limits_{x \to \pm\infty} f(x) = 0 \Rightarrow y = 0$ H.A.

② If $m = n$, then $\lim\limits_{x \to \pm\infty} f(x) = \dfrac{a_m}{b_n} \Rightarrow y = \dfrac{a_m}{b_n}$ H.A.

③ If $m > n$, then $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty \Rightarrow$ **no** H.A.

Ex 2 $\lim\limits_{x \to \infty} \dfrac{2 - 3x^3}{7 + 4x^3}$

3.3 (cont)

Ex 3 $\displaystyle\lim_{x\to\infty} \frac{4x^7-8x}{6x^4+9x^2}$

Ex 4 $\displaystyle\lim_{x\to\infty} \frac{5x+11}{7x^3-2}$

Ex 5 For $f(x)=\dfrac{x^2}{x+3}$, find

(a) $\displaystyle\lim_{x\to -3^-} f(x)$

(b) $\displaystyle\lim_{x\to -3^+} f(x)$

(c) $\displaystyle\lim_{x\to -3} f(x)$

3.3 (cont)

Ex 6 For $f(x) = \frac{x^2+x-2}{x+2}$, find

(a) $\lim_{x \to -2^-} f(x)$

(b) $\lim_{x \to -2^+} f(x)$

(c) $\lim_{x \to -2} f(x)$

Ex 7 Identify discontinuities & V.A.

(a) $f(x) = \frac{x^2+9}{9-x^2}$

(b) $f(x) = \frac{x^2+2x-3}{x^2-4x+3}$

3.3 (cont)

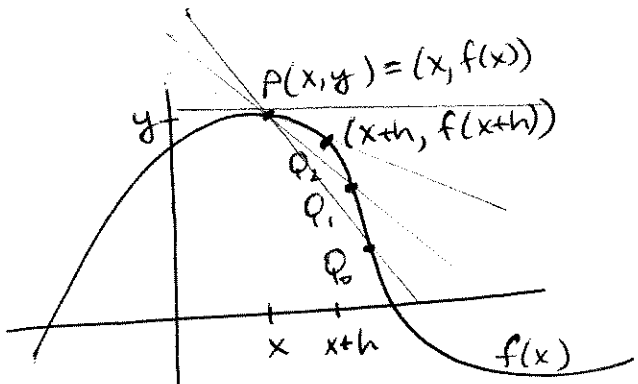
Ex 8 Find all H.A. + V.A.

(a) $f(x)=\dfrac{3x+2}{x-4}$

(b) $f(x)=\dfrac{x+5}{x^2}$

(c) $f(x)=\dfrac{x^2-x-12}{2x^2+5x-12}$

3.4 The Derivative



If we're looking for the slope of a curve at a point, we can estimate it by finding the slope of a <u>secant line</u>.

The actual slope at P will be the slope of the <u>tangent line</u>.

The slope of the secant line between P + Q_2 $\Rightarrow$ $m = \dfrac{\Delta y}{\Delta x}$

$$m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

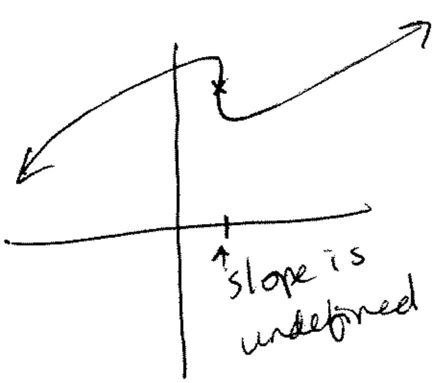
$\Rightarrow$ tangent line slope happens when h is arbitrarily small, i.e. as $h \to 0$.

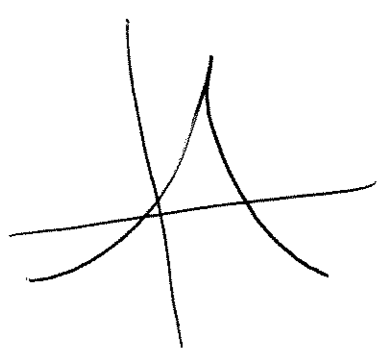
$$\boxed{\text{slope of tangent line} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)}$$

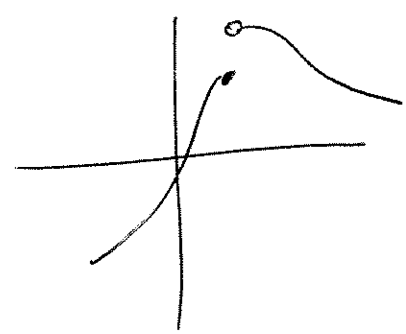
i.e. the slope of a function at a pt.

called the <u>derivative</u>

A function is differentiable if the derivative exists.







3.4 (cont)

Another perspective ⇒

Let's say an object moves along the y-axis so its location is $y = f(x) = x^2 + x$ at time x.

(y is in ft & x is in seconds)

What is the average velocity for x changing from 2 to 4 seconds?

$V_{avg} = ?$ for x going from 2 to 2.2 seconds?

$V_{avg} = ?$ for $x = 2$ to $x = 2+h$ seconds?

⇒ instantaneous velocity = ?

3.4 (cont)

Derivative (either as slope of curve, rate of change, or velocity)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiability $\Rightarrow$ continuity

(but continuity $\not\Rightarrow$ differentiability)

Ex 1 Use the defn of derivative to find $f'(x)$.

(a) $f(x) = -4x + 1$

(b) $f(x) = 3x^2 - 2$

Ex 2 Use defn of derivative to find $f'(x)$.

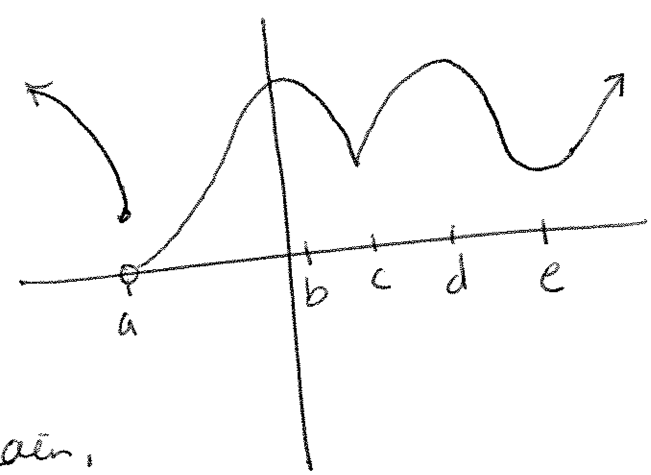
(a) $f(x) = x^3 + 5$

(b) $f(x) = 16\sqrt{x+9}$

3.4 (cont)

Ex 3 Find the equation of the tangent line to the graph of $f(x) = \frac{1}{x-1}$ at $(2,1)$.

Ex 4 Use this graph to answer these questions.



Does the derivative exist at

$x=a$?

$x=b$?

$x=c$?

$x=d$?

$x=e$?

Explain.

3.5 Basic Differentiation Rules (shortcuts ☺)

If $f(x) = c$ $(c \in \mathbb{R})$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$$

$\Rightarrow$ $\boxed{D_x[c] = 0 \quad \left(\text{or } \frac{d}{dx}(c) = 0\right)}$

The derivative of a constant is 0.

Let $f(x) = x^n$ $\quad n \in \mathbb{R}$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)x^{n-2}}{2}(h^2) + \dots + h^n - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} nx^{n-1} + h(\cdots \text{stuff} \cdots) = nx^{n-1}$$

$$\boxed{D_x[x^n] = nx^{n-1} \quad (n \in \mathbb{R})}$$ Power Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$$

$$\frac{d}{dx}[cf(x)] = c\,\frac{df(x)}{dx}$$

derivative is linear operator

3.5 (cont)

Ex 1 Find the derivative of each function.

(a) $f(x) = 2$

(b) $f(x) = 4x - 1$

(c) $f(t) = t^3 - 9t^2 + 2$

(d) $y = 2x^3 - x^2 + 3x - 1$

(e) $y = x^{5/2}$

(f) $g(x) = 4\sqrt[5]{x} + 2x$

(g) $s(t) = 3t^{-1} + t^{-2}$

3.5 (cont)

Ex 2 Find the derivative of each function.

(a) $y = \dfrac{3x}{x^{-5}}$

(b) $y = (x^2+2x)(x+1)$

(c) $f(x) = \dfrac{-6x^3+3x^2-2x+1}{x}$

Ex 3 Find the value of the derivative at the given pt.

(a) $f(x) = 3(5-x)^2$ at $(5,0)$

(b) $y = 3x\left(x^2 - \frac{2}{x}\right)$ at $(2,18)$

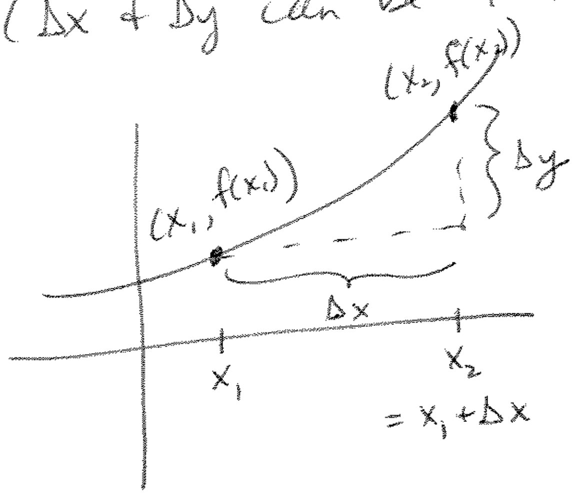
3.6 Differentials

Defn - Increments

For $y = f(x)$, $\Delta x = x_2 - x_1 \Rightarrow x_2 = x_1 + \Delta x$

$\Delta y = y_2 - y_1 = f(x_2) - f(x_1) = f(x_1 + \Delta x) - f(x_1)$

(Δx & Δy can be + or −)



$\Rightarrow \dfrac{\Delta y}{\Delta x} \simeq f'(x)$

$\Leftrightarrow \Delta y \simeq f'(x)\,\Delta x$

Defn - Differentials

If $y = f(x)$ is differentiable, then the differential dy is given by $dy = f'(x)\,dx$

($dx = \Delta x$).

$\Rightarrow \Delta y \simeq dy$

Ex 1 For $f(x) = 5x^2 - 2x$

(a) find Δx, given $x_1 = 1$, $x_2 = 3$

(b) Find Δy.

(c) Find $\dfrac{\Delta y}{\Delta x}$

3.6 (cont)

Ex 2 Find dy for each function.

(a) $y = 200x - \frac{x^2}{10}$

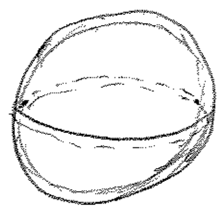
(b) $y = 5\sqrt{x}$

(c) $y = \frac{x^2}{(x+1)^2}$

Ex 3 For $y = 30 + 12x^2 - x^3$, $x = 2$, $dx = \Delta x = 0.1$, find dy + Δy.

3.6 (cont)

Ex 4 (#24) A sphere of radius 5 cm is coated w/ ice 0.1 cm thick. Use differentials to estimate the volume of the ice.



3.7 Marginal Analysis in Business + Economics

Defn If x = # units produced, then

$C(x)$ = total cost $\quad C'(x)$ = marginal cost

$R(x)$ = total revenue $\quad R'(x)$ = marginal revenue

$P(x) = R(x) - C(x)$ = total profit $\quad P'(x)$ = marginal profit $= R'(x) - C'(x)$

Thm If $C(x)$ is total cost of producing x items, the marginal cost fn approximates the exact cost of producing the $(x+1)^{st}$ item.

$$C'(x) \simeq C(x+1) - C(x).$$

(similarly for revenue + profit.)

Defn Average Cost $\quad \overline{C}(x) = \dfrac{C(x)}{x} \quad$ marginal avg cost $\overline{C}'(x)$

average revenue $\quad \overline{R}(x) = \dfrac{R(x)}{x} \quad$ marginal avg revenue $\overline{R}'(x)$

average profit $\quad \overline{P}(x) = \dfrac{P(x)}{x} \quad$ marginal avg profit $\overline{P}'(x)$

3.7 (cont)

Ex 1 Find the marginal revenue for
$R(x) = 50(20x - x^{3/2})$

Ex 2 The revenue from renting x apartments is given by $R(x) = 2x(900 + 32x - x^2)$.
(a) Find the additional revenue when the # of rentals is increased from 14 to 15.
(b) Find the marginal revenue when $x = 14$.
(c) what is meaning of (a) & (b)?

3.7 (cont)

Ex 3 (#10) The total profit (in \$) from the sale of x charcoal grills is $P(x) = 20x - 0.02x^2 - 320$

$x \in [0, 1000]$.

(a) Find avg profit per grill if 40 grills are produced.

(b) Find marginal avg profit at a production level of 40 grills + interpret the results.

(c) Use results from (a) + (b) to estimate the avg profit per grill if 41 grills are produced.

3.7 (cont)

Ex 4 (#14) The price-demand eqn + the cost for production of t.v. sets are given by

$x = 9000 - 30p$ (price-demand)

$C(x) = 150{,}000 + 30x$ (cost)

where x = # t.v.s sold at price $\$p$.

(a) Express p as $f(x)$ + find its domain.

(b) find marginal cost.

(c) Find $R(x)$ + its domain. (Hint: $R = xp$)

(d) Find marginal revenue.

3.7 (cont)

Ex 4 (cont)

(e) Find $R'(3000)$ + $R'(6000)$ + interpret these quantities.

(f) graph $C(x)$ + $R(x)$ on same axes for $x \in [0, 9000]$.
Find break-even pts.

(g) find $P(x)$.

(h) find $P'(x)$.

(i) Find $P'(1{,}500)$ $P'(4{,}500)$ + interpret these quantities.