3.1 Limits

Definition of a limit of a function

If \( f(x) \) becomes arbitrarily close to a single number \( L \) as \( x \) approaches \( c \) from either side, then \( \lim_{x \to c} f(x) = L \) ("the limit of \( f(x) \) as \( x \) approaches \( c \) is \( L \)).

Properties of limits

1. \( \lim_{x \to c} b = b \)
2. \( \lim_{x \to c} x = c \)
3. \( \lim_{x \to c} x^n = c^n \) (if \( n \) is a positive integer)
4. \( \lim_{x \to c} \sqrt{x} = \sqrt{c} \)

\( b, c \in \mathbb{R} \quad n \in \mathbb{Z}^+ \)

5. \( \lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \)
6. \( \lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) \)
7. \( \lim_{x \to c} [f(x)g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) \)
8. \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \) (assuming \( \lim_{x \to c} g(x) \neq 0 \))
9. \( \lim_{x \to c} [f(x)]^n = [\lim_{x \to c} f(x)]^n \)
10. \( \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} \) (if \( n \) is even, \( \lim_{x \to c} f(x) \) must be positive)
Ex 1 Find the limits

(a) \( \lim_{x \to 1} (2x^2 - 3x + 1) \)

(b) \( \lim_{x \to -2} \frac{3x+1}{2-x} \)

(c) \( \lim_{x \to 5} \frac{\sqrt{x+4} - 2}{5} \)

\textbf{Indeterminate Form}

If \( \lim f(x) = 0 \) and \( \lim g(x) = 0 \),
then \( \lim \frac{f(x)}{g(x)} \) is indeterminate,
(i.e. the \( \frac{0}{0} \) case)

(This means there's more to do!)
Ex 2 Find the limits.

(a) \( \lim_{x \to 3} \frac{3}{x-3} \)

(b) \( \lim_{x \to 3} \frac{2x^2-15}{x-3} \)

(c) \( \lim_{x \to 0} \frac{\sqrt{x+9} - 3}{x} \)

(d) \( \lim_{x \to 1} \frac{x+5}{x-1} \)
3.1 (cont)

One-sided limits

LH limit

\( \lim_{x \to c^-} f(x) = L \)

RH limit

\( \lim_{x \to c^+} f(x) = K \)

Ex 3

(a) \( \lim_{x \to 1^-} \frac{|3x-3|}{x-1} \)

(b) \( \lim_{x \to 1^+} \frac{|3x-3|}{x-1} \)
3.1 (cont)

If \( f \) is a function and \( c, L \in \mathbb{R} \), then

\[
\lim_{x \to c} f(x) = L \quad \text{iff} \quad \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L.
\]

**Ex 4** For \( f(x) = \begin{cases} 2x^2 - 1 & x > 0 \\ 3x - 1 & x < 0 \end{cases} \)

And \( \lim_{x \to 0} f(x) \).

**Ex 5** (a) \( \lim_{x \to 3} \frac{5}{x - 3} \)

(b) \( \lim_{x \to 3^-} \frac{5}{x - 3} \)

(c) \( \lim_{x \to 3^+} \frac{5}{x - 3} \)
Ex 6 \[ \lim_{{x \to 1^-}} \frac{2}{{x^2 - 1}} \]

\[ \lim_{{x \to 1^+}} \frac{2}{{x^2 - 1}} \]

\[ \lim_{{x \to 1}} \frac{2}{{x^2 - 1}} \]
3.2 Continuity

Defn of Continuity
A function \( f(x) \) is continuous at \( x = c \) if \( \lim_{x \to c} f(x) = f(c) \).
That is, \( f(c) \) must exist
\( \lim_{x \to c} f(x) \) must exist
and they have to be equal.

- All polynomials are continuous everywhere.
- All rational functions are continuous everywhere except where the denominator is zero.
3.2 (cont)

Ex 1 Determine where these functions are continuous.

(a) \( f(x) = 3x^3 - 4x^2 + x + 5 \)

(b) \( f(x) = \frac{x + 4}{x^2 - 6x + 5} \)

(c) \( f(x) = \frac{2}{x^2 - 1} \)

(d) \( f(x) = \begin{cases} x^2 - 4 & x \leq 0 \\ 3x + 1 & x > 0 \end{cases} \)

(e) \( h(x) = f(g(x)) \), \( f(x) = \frac{1}{x - 1} \), \( g(x) = x^2 + 5 \)

(f) \( g(x) = \sqrt{5 - x} \)
3.2 (cont)

Ex 2. Describe where \( f(x) \) is continuous.
\[
f(x) = x - \lfloor x \rfloor
\]

Ex 3. (\#38) Discuss continuity of
\[
f(x) = \frac{x}{x^2 - 4x + 3}
\]
on \([0,4]\). If there are any discontinuities, determine if it's removable.
3.3 Infinite limits & limits at Infinity

**Infinite limits & Vertical Asymptotes**

The vertical line \( x = a \) is a vertical asymptote for the graph of \( y = f(x) \) if

\[
f(x) \to \pm \infty \text{ as } x \to a^+ \text{ or } x \to a^-.
\]

**V.A. of Rational Fns**

If \( f(x) = \frac{n(x)}{d(x)} \) is a rational function, \( d(c) = 0 \) and \( n(c) \neq 0 \), then \( x = c \) is a V.A. of \( f \).

**Ex 1.** Find vertical asymptotes for

\[
f(x) = \frac{x^2 + 4}{(x-1)(x+3)}
\]
3, 3 (cont)

(1) If \( p \) is a positive \( R^+ \) and \( k \in R \), then
\[
\lim_{x \to \pm \infty} \frac{k}{x^p} = 0 \quad \text{and} \quad \lim_{x \to \pm \infty} kx^p = \pm \infty
\]

Limits of Rational Functions at \( \pm \infty \) and Rational Asymptotes

(A) If \( f(x) = \frac{a_n x^m + a_{n-1} x^{m-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0} \)
\( a_n \neq 0, b_n \neq 0 \)

Then
\[
\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{a_n x^m}{b_n x^n}.
\]

(B) H.A. (aka Horizontal Asymptotes)

3 cases

1. If \( m < n \), then \( \lim_{x \to \pm \infty} f(x) = 0 \) \( \Rightarrow \) \( y = 0 \) H.A.
2. If \( m = n \), then \( \lim_{x \to \pm \infty} f(x) = \frac{a_n}{b_n} \) \( \Rightarrow \) \( y = \frac{a_n}{b_n} \) H.A.
3. If \( m > n \), then \( \lim_{x \to \pm \infty} f(x) = \pm \infty \) \( \Rightarrow \) no H.A.

Ex 2
\[
\lim_{x \to \infty} \frac{2-3x^3}{7+4x^3}
\]
3.3 (cont)

Ex 3 \( \lim_{x \to \infty} \frac{4x^2 - 8x}{6x^4 + 9x^2} \)

Ex 4 \( \lim_{x \to \infty} \frac{5x + 11}{7x^3 - 2} \)

Ex 5 For \( f(x) = \frac{x^2}{x+3} \), find

(a) \( \lim_{x \to -3^-} f(x) \)

(b) \( \lim_{x \to -3^+} f(x) \)

(c) \( \lim_{x \to 3} f(x) \)
3.3 (cont.)

Example 6
For \( f(x) = \frac{x^2 + x - 2}{x - 2} \), find

(a) \( \lim_{x \to -2^-} f(x) \)

(b) \( \lim_{x \to -2^+} f(x) \)

(c) \( \lim_{x \to -2} f(x) \)

Example 7
Identify discontinuities and V.A.

(a) \( f(x) = \frac{x^2 + 9}{9 - x^2} \)

(b) \( f(x) = \frac{x^2 + 2x - 3}{x^2 - 4x + 3} \)
3.3 (cont.)

Ex. 8 Find all H.A. + V.A.

(a) \( f(x) = \frac{3x+2}{x-4} \)

(b) \( f(x) = \frac{x+5}{x^2} \)

(c) \( f(x) = \frac{x^2-x-12}{2x^2+5x-12} \)
If we're looking for the slope of a curve at a point, we can estimate it by finding the slope of a secant line. The actual slope at P will be the slope of the **tangent line**.

The slope of the secant line between P and Q₂ is:

\[ m = \frac{Δy}{Δx} \]

\[ m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h} \]

\[ \Rightarrow \text{tangent line slope happens when } h \text{ is arbitrarily small, i.e. as } h \to 0. \]

\[ \text{slope of tangent line } = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \]

i.e. the slope of a function at a pt.

A function is differentiable if the derivative exists.
3.4 (cont)

Another perspective

Let's say an object moves along the y-axis so its location is \( y = f(x) = x^2 + x \) at time \( x \) (\( y \) is in ft and \( x \) is in seconds).

What is the average velocity for \( x \) changing from 2 to 4 seconds?

\[ \text{Vavg} \] for \( x \) going from 2 to 2.2 seconds?

\[ \text{Vavg} \] for \( x = 2 \) to \( x = 2 \text{th} \) seconds?

\[ \Rightarrow \text{instantaneous velocity} = \ ? \]
3.4 (cont)

**Derivative**

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

Differentiability \(\Rightarrow\) continuity

(but continuity \(\not\Rightarrow\) differentiability)

**Ex1** Use the defn of derivative to find \(f'(x)\).

(a) \(f(x) = -4x + 1\)

(b) \(f(x) = 3x^2 - 2\)
3.4 (cont)

Ex 2 Use defn of derivative to find $f'(x)$.

(a) $f(x) = x^3 + 5$

(b) $f(x) = 16\sqrt{x+9}$
3.4 (cont)

Ex 3 Find the equation of the tangent line to the graph of \( f(x) = \frac{1}{x-1} \) at \((2,1)\).

Ex 4 Use this graph to answer these questions.

Does the derivative exist at:

- \( x = a \)?
- \( x = b \)?
- \( x = c \)?
- \( x = d \)?
- \( x = e \)?

Explain.
3.5 Differentiation Rules (Shortcuts 🤖)

If \( f(x) = c \) (\( c \in \mathbb{R} \)), then

\[
\frac{f'(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0
\]

\( \Rightarrow \quad D_x[c] = 0 \) \( (\text{or } \frac{d}{dx}(c) = 0) \)

The derivative of a constant is 0.

Let \( f(x) = x^n \), \( n \in \mathbb{R} \)

\[
\Rightarrow f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)x^{n-2}h^2}{2} + \ldots + h^n - x^n}{h}
\]

\[
= \lim_{h \to 0} nx^{n-1} + \text{other terms} = nx^{n-1}
\]

\( \Rightarrow \quad D_x[x^n] = nx^{n-1} \) \( (n \in \mathbb{R}) \)

\[
\frac{d}{dx} [f(x) \pm g(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}
\]

\[
\frac{d}{dx} [cf(x)] = c \frac{df(x)}{dx}
\]

\( \text{derivative is linear operator} \)

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3.5 (cont)

Example 1: Find the derivative of each function.

(a) $f(x) = 2$

(b) $f(x) = 4x - 1$

(c) $f(t) = t^3 - 9t^2 + 2$

(d) $y = 2x^3 - x^2 + 3x - 1$

(e) $y = x^{5/2}$

(f) $g(x) = 4\sqrt[5]{x} + 2x$

(g) $s(t) = 3t^{-1} + t^{-2}$
3.5 (cont)

Ex 2 Find the derivative of each function.

(a) \( y = \frac{3x}{x^2 - 3} \)

(b) \( y = (x^2 + 2x)(x+1) \)

(c) \( f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x} \)

Ex 3 Find the value of the derivative at the given pt.

(a) \( f(x) = 3(5-x)^2 \) at \((5,0)\)

(b) \( y = 3x(x^2 - \frac{2}{x}) \) at \((2,18)\)
3.6 Differentials

**Defn - Increments**

For \( y = f(x) \),

\[ \Delta x = x_2 - x_1 \quad \Rightarrow \quad x_2 = x_1 + \Delta x \]

\[ \Delta y = y_2 - y_1 = f(x_2) - f(x_1) = f(x_1 + \Delta x) - f(x_1) \]

\((\Delta x + \Delta y) \text{ can be } \alpha + \omega -\)

**Ex 1** For \( f(x) = 5x^2 - 2x \)

(a) Find \( \Delta x \), given \( x_1 = 1 \) \( x_2 = 3 \)

(b) Find \( \Delta y \).

(c) Find \( \frac{\Delta y}{\Delta x} \)

**Defn - Differentials**

If \( y = f(x) \) is differentiable, then the differential \( dy \) is given by \( dy = f'(x) \, dx \)

\((dx = \Delta x)\),

\( \Rightarrow \quad dy \approx dy \)
3.6 (cont)

Ex 2 Find $dy$ for each function.

(a) $y = 200x - \frac{x^2}{10}$

(b) $y = 5\sqrt{x}$

(c) $y = \frac{x^2}{(x+1)^2}$

Ex 3 For $y = 30 + 12x^2 - x^3$, $x = 2$, $dx = \Delta x = 0.1$, find $dy + \Delta y$. 

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Ex 4 (24) A sphere of radius 5 cm is
covered by ice 0.1 cm thick. Use differentials
to estimate the volume of the ice.
3.7 Marginal Analysis in Business & Economics

**Defn** If \( x \) = # units produced, then
\[
\begin{align*}
C(x) &= \text{total cost} \quad C'(x) = \text{marginal cost} \\
R(x) &= \text{total revenue} \quad R'(x) = \text{marginal revenue} \\
P(x) &= R(x) - C(x) = \text{total profit} \quad P'(x) = \text{marginal profit} \\
&= R'(x) - C'(x)
\end{align*}
\]

Thus, if \( C(x) \) is total cost of producing \( x \) items, the marginal cost for approximates the exact cost of producing the \((x+1)st\) item.

\[
C'(x) \approx C(x+1) - C(x).
\]

(Similarly for revenue & profit.)

**Defn** Average Cost \( \overline{C}(x) = \frac{C(x)}{x} \) \quad \text{marginal avg. cost} \ \overline{C}'(x)

average revenue \( \overline{R}(x) = \frac{R(x)}{x} \) \quad \text{marginal avg. revenue} \ \overline{R}'(x)

average profit \( \overline{P}(x) = \frac{P(x)}{x} \) \quad \text{marginal avg. profit} \ \overline{P}'(x)

\[\text{Math1500}\]
3.7 (cont)

**Ex 1.** Find the marginal revenue for

\[ R(x) = 50(20x - x^3) \]

**Ex 2.** The revenue from renting \( x \) apartments is given by \( R(x) = 2x(900 + 32x - x^2) \).

(a) Find the additional revenue when the # of rentals is increased from 14 to 15.

(b) Find the marginal revenue when \( x = 14 \).

(c) What is meaning of (a) + (b)?
3.7 (cont)

Ex 3. The total profit (in $) from the sale of $10 \times$ charcoal grills is \( P(x) = 20x - 0.02x^2 - 320 \) \( x \in [0,1000] \).

(a) Find the profit per grill if 40 grills are produced.

(b) Find the marginal profit at a production level of 40 grills and interpret the results.

(c) Use results from (a) + (b) to estimate the profit per grill if 41 grills are produced.
3.7 (cont)

Ex. 4 (#14) The price-demand equ. the cost for
for production of tv. sets are given by
x = 9000 - 30p    (price-demand)
C(x) = 150000 + 30x    (cost)    where x = # tv.s
sold at price $p.

(a) Express p as a function and find its domain.

(b) Find marginal cost.

(c) Find R(x) = its domain. (hint: R=xp)

(d) Find marginal revenue.
Ex 4 (cont)

(e) Find \( R'(3000) \) and \( R'(6000) \) and interpret these quantities.

(f) graph \( C(x) + P(x) \) on same axes for \( x \in [0,9000] \). Find break-even pts.

(g) find \( P(x) \).

(h) find \( P'(x) \).

(i) Find \( P'(1500) \) and \( P'(4500) \) and interpret these quantities.