5.1 Area of Plane Region

Ex 1: Find the area between these curves.
\[ y = \sqrt{x}, \quad y = x - 4, \quad x = 0 \]

Area between curves = \[ \int_a^b (f(x) - g(x)) \, dx \]

Area of region between \( f \) and \( g \) = Area of region under \( f(x) \) - Area of region under \( g(x) \)

Area between curves = \[ \int_c^d (f(y) - g(y)) \, dy \]

Area between curves = \[ \int_a^b (f(x) - g(x)) \, dx \]
5.1 (continued)

Ex 2: Find the area between these curves.
\[ x = (3 - y)(y + 1), \quad x = 0 \]

Ex 3: Find the area between these curves.
\[ y = (x - 3)(x - 1), \quad y = x \]

Ex 4: Find the area between these curves.
\[ x = 4y^4, \quad x = 8 - 4y^2 \]
### 5.2 & 5.3 Volumes of Solids of Revolution

<table>
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<tr>
<th>Disk Method</th>
<th>Washer Method</th>
<th>Shell Method</th>
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</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Disk Method Diagram" /></td>
<td><img src="image2.png" alt="Washer Method Diagram" /></td>
<td><img src="image3.png" alt="Shell Method Diagram" /></td>
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</table>

#### Disk Method
- Calculated volume using the disk method by integrating the area of disks formed by rotating a cross-section around an axis.

#### Washer Method
- Used when there is a gap or a hole in the solid of revolution, integrating the difference between two functions.

#### Shell Method
-适用于旋转体的体积计算，通过将旋转体分解为壳体，然后计算壳体的体积积分。
Ex 1: Find the volume of the solid generated by the indicated region being revolved about the given axis.

(a) \( y = \frac{x^{2/3}}{3}, y = 0, x = -2, x = 3 \) about the x-axis

**Disk Method**

\[
V = \pi \int_{a}^{b} r^2 \, dx \quad \text{or} \quad dy
\]

where

- \( r = \) radius of disk

**Washer Method**

\[
V = \pi \int_{a}^{b} (r_{\text{outer}}^2 - r_{\text{inner}}^2) \, dx \quad \text{or} \quad dy
\]

where

- \( r_{\text{outer}} = \) outer radius of washer
- \( r_{\text{inner}} = \) inner radius of washer

(b) \( y = \frac{x^{2/3}}{3}, y = 0, x = -2, x = 3 \) about the line \( y = -1 \)

**Shell Method**

\[
V = 2\pi \int_{a}^{b} r \, h \, dx \quad \text{or} \quad dy
\]

where

- \( r = \) radius of shell
- \( h = \) height of shell

(c) \( y = \frac{x^{2/3}}{3}, y = 0, x = -2, x = 3 \) about the line \( y = -4 \)

**When to use \( dx \) or \( dy \)?**

<table>
<thead>
<tr>
<th>rotate about horizontal line</th>
<th>rotate about vertical line</th>
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<tbody>
<tr>
<td>( dx )</td>
<td>( dy )</td>
</tr>
<tr>
<td>( dy )</td>
<td>( dx )</td>
</tr>
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</table>
Ex 2: Set up the volume integrals for the region bounded by the curves
\[ x^2 + y^2 = 4, \ y = 0, \ x = 0, \ x = 1 \]
(a) rotated about the x-axis.
(b) rotated about the y-axis.
(c) rotated about the line \( x = 2 \).

Ex 3: Set up the volume integrals for the region bounded by the curves
\[ y = -2x^2 + 4x + 3, \ y = 3 \]
(a) rotated about the y-axis.
(b) rotated about the x-axis.
(c) rotated about the line \( y = -1 \).
Ex 4: (#19 from book) A round hole of radius $a$ is drilled through the center of a solid sphere of radius $b$ (such that $b > a$). Find the volume of the remaining solid.
5.4 Length of a Curve/Surface Area

Ex 1: Find the length of the indicated curve.
(a) \(30x^3 - y^8 = 15\) between \(y = 1\) and \(y = 3\)

In general,
\[ L = \int_a^b ds \quad \text{where } ds = \text{a little bit of arc length} \]

\[ ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

OR

\[ ds = \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \]

OR

\[ ds = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]

(b) \(x = a \cos t + a t \sin t\)
and \(y = a \sin t - a t \cos t\), \(t \in [-1, 1]\)
(Assume \(a\) is a constant.)
5.4 (continued)

Ex 2: Find the surface area of the surface created when you revolve 

\[ y = \frac{x^6 + 2}{8x^2}, \quad x \in [1, 3] \]

about the x-axis.

Surface Area:

The surface area of the surface created by rotating the curve \( y = f(x) \) about the x-axis is given by

\[
\text{SA} = \int_a^b 2\pi f(x) \, ds
\]

where

\[
ds = \sqrt{1 + (f'(x))^2} \, dx
\]

Ex 3: Show that the area of the part of the surface of a sphere of radius \( a \) between two parallel planes \( h \) units apart \( (h < 2a) \) is \( 2\pi ah \).
Ex 1: For a certain type of nonlinear spring, the force required to keep the spring stretched a distance \( x \) is given by \( F(x) = k x^{4/3} \). If the force required to keep it stretched 8 inches is 2 pounds, how much work is done in stretching this spring 27 inches?

Work:

\[
W = \int_{a}^{b} F(x) \, dx
\]

where \( F(x) \) is a force.

Ex 2: A 10-pound monkey hangs at the end of a 20-foot chain that weighs 0.5 pound/foot. How much work does it do in climbing the chain to the top? (Assume the end of the chain is attached to the monkey.)
5.6 Moments and Center of Mass

Ex 1: Find the centroid of the region bounded by the given curves.

(a) \( y = x^2 \), \( y = 2x + 3 \)

mass \( m = \delta \int_a^b (f(x) - g(x)) \, dx \)

\( M_y = \delta \int_a^b x(f(x) - g(x)) \, dx \)

\( M_x = \frac{\delta}{2} \int_a^b [(f(x))^2 - (g(x))^2] \, dx \)

\( \bar{x} = \frac{M_y}{m} \), \( \bar{y} = \frac{M_x}{m} \) where

\((\bar{x}, \bar{y}) = \text{center of mass (or centroid)} \)

for a homogeneous lamina

(b) \( y = x^2 \), \( y = 4x \)