

Derivatives

$$\begin{aligned}
D_x e^x &= e^x \\
D_x \sin(x) &= \cos(x) \\
D_x \cos(x) &= -\sin(x) \\
D_x \tan(x) &= \sec^2(x) \\
D_x \cot(x) &= -\csc^2(x) \\
D_x \sec(x) &= \sec(x)\tan(x) \\
D_x \csc(x) &= -\csc(x)\cot(x) \\
D_x \sin^{-1}(x) &= \frac{1}{\sqrt{1-x^2}}, x \in (-1, 1) \\
D_x \cos^{-1}(x) &= \frac{-1}{\sqrt{1-x^2}}, x \in (-1, 1) \\
D_x \tan^{-1}(x) &= \frac{1}{1+x^2}, x \in \mathbb{R} \\
D_x \sec^{-1}(x) &= \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1 \\
D_x \sinh(x) &= \cosh(x) \\
D_x \cosh(x) &= \sinh(x) \\
D_x \tanh(x) &= \text{sech}^2(x) \\
D_x \coth(x) &= -\text{csch}^2(x) \\
D_x \text{sech}(x) &= -\text{sech}(x)\tanh(x) \\
D_x \text{csch}(x) &= -\text{csch}(x)\coth(x) \\
D_x \sinh^{-1}(x) &= \frac{1}{\sqrt{x^2+1}} \\
D_x \cosh^{-1}(x) &= \frac{1}{\sqrt{x^2-1}}, x > 1 \\
D_x \tanh^{-1}(x) &= \frac{1}{1-x^2}, -1 < x < 1 \\
D_x \text{sech}^{-1}(x) &= \frac{1}{x\sqrt{1-x^2}}, 0 < x < 1 \\
D_x \ln(x) &= \frac{1}{x}
\end{aligned}$$

Integrals

$$\begin{aligned}
\int \frac{1}{x} dx &= \ln|x| + c \\
\int e^x dx &= e^x + c \\
\int a^x dx &= \frac{a^x}{\ln a} + c \\
\int e^{ax} dx &= \frac{1}{a} e^{ax} + c \\
\int \sqrt{1-x^2} dx &= \sin^{-1}(x) + c \\
\int \frac{1}{1+x^2} dx &= \tan^{-1}(x) + c \\
\int \frac{1}{\sqrt{a^2-x^2}} dx &= \sin^{-1}\left(\frac{x}{a}\right) + c \\
\int \sinh(x) dx &= \cosh(x) + c \\
\int \cosh(x) dx &= \sinh(x) + c \\
\int \tanh(x) dx &= \ln|\cosh(x)| + c \\
\int \tanh(x) \text{sech}(x) dx &= -\text{sech}(x) + c \\
\int \text{sech}^2(x) dx &= \tanh(x) + c \\
\int \text{csch}(x) \coth(x) dx &= -\text{csch}(x) + c \\
\int \tan(x) dx &= -\ln|\cos(x)| + c \\
\int \cot(x) dx &= \ln|\sin(x)| + c \\
\int \cos(x) dx &= \sin(x) + c \\
\int \sin(x) dx &= -\cos(x) + c \\
\int \sqrt{a^2-u^2} du &= \frac{1}{2} \sin^{-1}\left(\frac{u}{a}\right) + c \\
\int \frac{1}{1+u^2} du &= \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + c \\
\int \ln(x) dx &= x \ln(x) - x + c
\end{aligned}$$

U-Substitution

Let $u = f(x)$ (be more than one variable)
 Determine: $du = \frac{df(x)}{dx} dx$ and solve for dx .
 Then, if a definite integral, substitute the bounds for $u = f(x)$ at each bound.
 Solve the integral using u .

Integration by Parts

$$\int u dv = uv - \int v du$$

Fns and Identities

$$\begin{aligned}
\sin(\cos^{-1}(x)) &= \sqrt{1-x^2} \\
\cos(\sin^{-1}(x)) &= \sqrt{1-x^2}
\end{aligned}$$

Directional Derivatives

Let $z=f(x,y)$, be a function, (a,b) apoint in the domain (a valid input point) and \hat{u} a unit vector (2D).
 The Directional Derivative is then the derivative at the point (a,b) in the direction of \hat{u} or:
 $D_{\hat{u}} f(a,b) = \hat{u} \cdot \nabla f(a,b)$
 This will return a scalar. 4-D version:
 $D_{\hat{u}} f(a,b,c) = \hat{u} \cdot \nabla f(a,b,c)$

Tangent Planes

Let $F(x,y,z) = k$ be a surface and $P = (x_0, y_0, z_0)$ be a point on that surface. Equation of a Tangent Plane:
 $\nabla F(x_0, y_0, z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

Approximations

Let $z = f(x,y)$ be a differentiable function total differential of $f = dz$
 $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$
 This is the approximate change in z
 The actual change in z is the difference in z values:
 $\Delta z = z - z_1$

Maxima and Minima

Internal Points
 1. Take the Partial Derivatives with respect to X and Y (f_x and f_y) (Can use gradient).
 2. Set derivatives equal to 0 and use to solve system of equations for x and y .
 3. Plug back into original equation for z .
 Use Second Derivative Test for whether points are local max, min, or saddle.

Second Partial Derivative Test

1. Find all (x,y) points such that $\nabla f(x,y) = 0$
 2. Let $D = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2(x,y)$
 IF (a) $D > 0$ AND $f_{xx} < 0$, $f(x,y)$ is local max value.
 (b) $D > 0$ AND $f_{xx}(x,y) > 0$ $f(x,y)$ is local min value.
 (c) $D < 0$, $(x,y,f(x,y))$ is a saddle point.
 (d) $D = 0$, test is inconclusive.
 3. Determine if any boundary point gives min or max. Typically, we have to parametrize boundary and then reduce to a Calc 1 type of min/max problem to solve.
The following only apply if a boundary is given
 1. check the corner points
 2. Check each line ($0 \leq x \leq 5$ would give $x=0$ and $x=5$)
 On Bounded Equations, this is the global min and max...second derivative test is not needed.

Lagrange Multipliers

Given a function $f(x,y)$ with a constraint $g(x,y)$, solve the following system of equations to find the max and min points on the constraint (NOTE: may need to also find internal points).
 $\nabla f = \lambda \nabla g$
 $g(x,y) = 0$ (if applicable)

$$\begin{aligned}
\sec(\tan^{-1}(x)) &= \sqrt{1+x^2} \\
\tan(\sec^{-1}(x)) &= \frac{x}{\sqrt{x^2-1}} \\
&= \sqrt{x^2-1} \text{ if } x \geq 1 \\
&= -\sqrt{x^2-1} \text{ if } x \leq -1 \\
\sinh^{-1}(x) &= \ln|x + \sqrt{x^2+1}| \\
\sinh^{-1}(x) &= \ln|x + \sqrt{x^2-1}|, x \geq -1 \\
\sinh^{-1}(x) &= \frac{1}{2} \ln|x + \frac{1+x}{1-x}|, -1 < x < 1 \\
\text{sech}^{-1}(x) &= \ln\left|\frac{1+\sqrt{1-x^2}}{1-x}\right|, 0 < x \leq 1 \\
\sinh(x) &= \frac{e^x - e^{-x}}{2} \\
\cosh(x) &= \frac{e^x + e^{-x}}{2}
\end{aligned}$$

Trig Identities

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
1 + \tan^2(x) &= \sec^2(x) \\
1 + \cot^2(x) &= \csc^2(x) \\
\sin(x \pm y) &= \sin(x)\cos(y) \pm \cos(x)\sin(y) \\
\cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y) \\
\tan(x \pm y) &= \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)} \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
\cosh(n^2 x) - \sinh^2 x &= 1 \\
1 + \tan^2(x) &= \sec^2(x) \\
1 + \cot^2(x) &= \csc^2(x) \\
\sin^2(x) &= \frac{1-\cos(2x)}{2} \\
\cos^2(x) &= \frac{1+\cos(2x)}{2} \\
\tan^2(x) &= \frac{1-\cos(2x)}{1+\cos(2x)} \\
\sin(-x) &= -\sin(x) \\
\cos(-x) &= \cos(x) \\
\tan(-x) &= -\tan(x)
\end{aligned}$$

Calculus 3 Concepts

Cartesian coords in 3D
 given two points:
 (x_1, y_1, z_1) and (x_2, y_2, z_2) .
 Distance between them:
 $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$
 Midpoint:
 $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$
 Sphere with center, (h,k,l) and radius r:
 $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Vectors

Vector: \vec{u}
 Unit Vector: \hat{u}
 Magnitude: $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$
 Unit Vector: $\hat{u} = \frac{\vec{u}}{|\vec{u}|}$
Dot Product
 $\vec{u} \cdot \vec{v}$
 Produces a Scalar
 (Geometrically, the dot product is a vector projection)
 $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$
 $\vec{u} \cdot \vec{v} = 0$ means the two vectors are perpendicular θ is the angle between them
 $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos(\theta)$
 $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$
 NOTE:
 $\vec{u} \cdot \hat{v} = \cos(\theta)$
 $|\vec{u}| |\hat{v}| = \vec{u} \cdot \hat{v}$
 $\vec{u} \cdot \vec{v} = 0$ when \perp
 Angle between \vec{u} and \vec{v} :
 $\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$

Double Integrals

With respect to the xy-axis, if taking an integral.
 $\int \int_D f(x,y) dx dy$ is cutting in vertical rectangles,
 $\int \int_R f(x,y) dy dx$ is cutting in horizontal rectangles
Polar Coordinates
 When using polar coordinates,
 $dA = r dr d\theta$
Surface Area of a Curve
 Let $z = f(x,y)$ be continuous over S (a closed region in 2D domain)
 Then the surface area of $z = f(x,y)$ over S is:
 $SA = \int \int_S \sqrt{f_x^2 + f_y^2 + 1} dA$

Triple Integrals

$\int \int \int_V f(x,y,z) dv = \int_{c_1}^{c_2} \int_{b_1(x)}^{b_2(x,y)} \int_{a_1(x,y)}^{a_2(x,y,z)} f(x,y,z) dz dy dx$
 Note: dv can be exchanged for $dz dy dx$ in any order, but you must then choose your limits of integration according to that order

Jacobian Method

$\int \int_G f(u,v) |J(u,v)| du dv = \int \int_R f(x,y) dx dy$
 $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$
 Common Jacobians:
 Rect. to Cylindrical: r
 Rect. to Spherical: $\rho^2 \sin(\phi)$

Vector Fields

let $f(x,y,z)$ be a scalar field and $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ be a vector field.
 Gradient of $f = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$
 Divergence of \vec{F} :
 $\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$
 Curl of \vec{F} :
 $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

Line Integrals

C given by $x = x(t), y = y(t), t \in [a,b]$
 $\int_C f(x,y) ds = \int_a^b f(x(t), y(t)) |ds|$
 where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
 or $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$
 or $\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$
 To evaluate a Line Integral,
 - get a parameterized version of the line (usually in terms of t , though in exclusive terms of x or y is ok)
 - evaluate for the derivatives needed (usually $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$)
 - plug in to original equation to get in terms of the independent variable
 - solve integral

Projection of \vec{u} onto \vec{v} :
 $pr_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right) \vec{v}$
Cross Product
 $\vec{u} \times \vec{v}$
 Produces a Vector
 (Geometrically, the cross product is the area of a parallelogram with sides $|\vec{u}|$ and $|\vec{v}|$)
 $\vec{u} = \langle u_1, u_2, u_3 \rangle$
 $\vec{v} = \langle v_1, v_2, v_3 \rangle$
 $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$\vec{u} \times \vec{v} = \vec{0}$ means the vectors are parallel

Lines and Planes

Equation of a Plane
 (x_0, y_0, z_0) is a point on the plane and $\langle A, B, C \rangle$ is a normal vector
 $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$
 $A, B, C > 0, \langle x-x_0, y-y_0, z-z_0 \rangle > 0$
 $Ax + By + Cz = D$ where
 $D = Ax_0 + By_0 + Cz_0$

Equation of a line
 A line requires a Direction Vector
 $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and a point (x_1, y_1, z_1)
 then,
 a parameterization of a line could be:
 $x = u_1 t + x_1$
 $y = u_2 t + y_1$
 $z = u_3 t + z_1$

Distance from a Point to a Plane
 The distance from a point (x_0, y_0, z_0) to a plane $Ax+By+Cz=D$ can be expressed by the formula:
 $d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$

Coord Sys Conv

Cylindrical to Rectangular
 $x = r \cos(\theta)$
 $y = r \sin(\theta)$
 $z = z$
Rectangular to Cylindrical
 $r = \sqrt{x^2 + y^2}$
 $\tan(\theta) = \frac{y}{x}$
 $z = z$
Spherical to Rectangular
 $x = \rho \sin(\phi) \cos(\theta)$
 $y = \rho \sin(\phi) \sin(\theta)$
 $z = \rho \cos(\phi)$
Rectangular to Spherical
 $\rho = \sqrt{x^2 + y^2 + z^2}$
 $\tan(\theta) = \frac{y}{x}$
 $\cos(\phi) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$
Spherical to Cylindrical
 $r = \rho \sin(\phi)$
 $\theta = \theta$
 $z = \rho \cos(\phi)$
Cylindrical to Spherical
 $\rho = \sqrt{r^2 + z^2}$
 $\theta = \theta$
 $\cos(\phi) = \frac{z}{\sqrt{r^2 + z^2}}$

Work
 Let $\vec{F} = M\hat{i} + \hat{j} + \hat{k}$ (force)
 $M = M(x,y,z), N = N(x,y,z), P = P(x,y,z)$
 (Literally) $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$
 Work $w = \int_C \vec{F} \cdot d\vec{r}$
 (Work done by moving a particle over curve C with force \vec{F})

Independence of Path

Fund Thm of Line Integrals
 C is curve given by $\vec{r}(t), t \in [a,b]$; $\vec{r}'(t)$ exists. If $f(\vec{r})$ is continuously differentiable on an open set containing C, then $\int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$
Equivalent Conditions
 \vec{F} continuous on open connected set D. Then,
 (a) $\vec{F} = \nabla f$ for some fn f . (if \vec{F} is conservative)
 \Leftrightarrow (b) $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is indep. of path in D.
 \Leftrightarrow (c) $\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = 0$ for all closed paths in D.

Conservation Theorem
 $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ continuously differentiable on open, simply connected set D.
 \vec{F} conservative $\Leftrightarrow \nabla \times \vec{F} = \vec{0}$
 (in 2D $\nabla \times \vec{F} = 0$ iff $M_y = N_x$)

Green's Theorem

(method of changing line integral for double integral - Use for Flux and Circulation across 2D curve and line integrals over a closed boundary)
 $\oint_C M dx - N dy = \iint_R (M_x - N_y) dx dy$
 $\oint_C M dx + N dy = \iint_R (N_x - M_y) dx dy$
 Let: R be a region in xy-plane
 $\cdot C$ is simple, closed curve enclosing R (w/ parameterization $\vec{r}(t)$)
 $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ be continuously differentiable over RUC.
Form 1: Flux Across Boundary
 \vec{n} = unit normal vector to C
 $\oint_C \vec{F} \cdot \vec{n} = \iint_R \nabla \cdot \vec{F} dA$
 $\Leftrightarrow \oint_C M dx - N dy = \iint_R (M_x - N_y) dx dy$
Form 2: Circulation Along Boundary
 $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \nabla \times \vec{F} \cdot \hat{u} dA$
 $\Leftrightarrow \oint_C M dx + N dy = \iint_R (N_x - M_y) dx dy$
Area of R
 $A = \oint_C \left(\frac{y}{2} dx - \frac{x}{2} dy\right)$

Gauss' Divergence Thm

(3D Analog of Green's Theorem - Use for Flux over a 3D surface) Let:
 $\vec{F}(x,y,z)$ be vector field continuously differentiable in solid S
 $\cdot S$ is a 3D solid ∂S boundary of S (A Surface)
 $\cdot \hat{n}$ unit normal to ∂S
 Then,
 $\oint_{\partial S} \vec{F}(x,y,z) \cdot \hat{n} dS = \iiint_S \nabla \cdot \vec{F} dV$
 $(dV = dx dy dz)$

Surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



Hyperboloid of Two Sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$



Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2 = 0$$



Cylinder

$$(x-a)^2 + (y-b)^2 = c^2$$



Partial Derivatives

Partial Derivatives are simply holding all other variables constant (and set like constants for the derivative) and only taking the derivative with respect to a given variable.

Surface Integrals

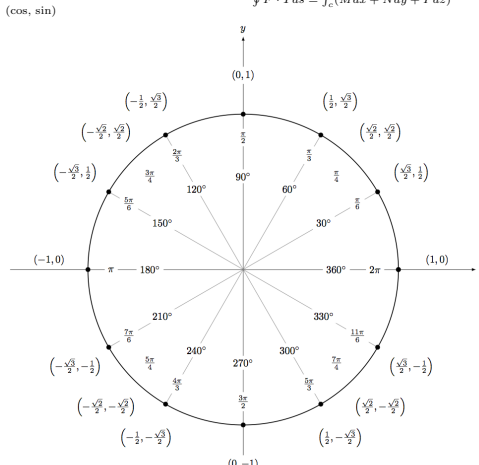
Let R be closed, bounded region in xy-plane
 $\cdot f$ be a fn with first order partial derivatives on R
 $\cdot G$ be a surface over R given by $z = f(x,y)$
 $\cdot g(x,y,z) = g(x,y,f(x,y))$ is cont. on R
 Then,
 $\iint_R g(x,y,z) dS = \iint_R g(x,y,f(x,y)) dS$
 where $dS = \sqrt{f_x^2 + f_y^2 + 1} dy dx$

Flux of F across G

$\iint_G \vec{F} \cdot \hat{n} dS = \iint_R -M dx - N dy + P dz dy dx$
 where:
 $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$
 $\cdot G$ is surface $f(x,y,z) = z$
 $\cdot \hat{n}$ is upward unit normal on G.
 $\cdot f(x,y)$ has continuous 1st order partial derivatives

Unit Circle

(cos, sin)



Given $z=f(x,y)$, the partial derivative of z with respect to x is:

$$\begin{aligned}
f_x(x,y) &= z_x = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x} \\
&\text{likewise for partial with respect to } y: \\
f_y(x,y) &= z_y = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y}
\end{aligned}$$

Notation
 For f_{xy} , work "inside to outside" f_x then f_{xy} , then f_{xyy}
 $f_{xyy} = \frac{\partial^3 f}{\partial x \partial y^2}$
 For $\frac{\partial^3 f}{\partial x^2 \partial y}$, work right to left in the denominator

Gradients

The Gradient of a function in 2 variables is $\nabla f = \langle f_x, f_y \rangle$
 The Gradient of a function in 3 variables is $\nabla f = \langle f_x, f_y, f_z \rangle$

Chain Rule(s)

Take the Partial derivative with respect to the first-order variables of the function times the partial (or normal) derivative of the first-order variable to the ultimate variable you are looking for summed with the same process for other first-order variables this makes sense here. Example:
 Let $x = x(s,t)$, $y = y(t)$ and $z = z(x,y)$.
 z then has first partial derivative:
 $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$
 Note: the use of d instead of ∂ with the function of only one independent variable

In this case (with z containing x and y as well as x and y both containing s and t), the chain rule for $\frac{\partial z}{\partial s}$ is $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$
 The chain rule for $\frac{\partial z}{\partial t}$ is $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$