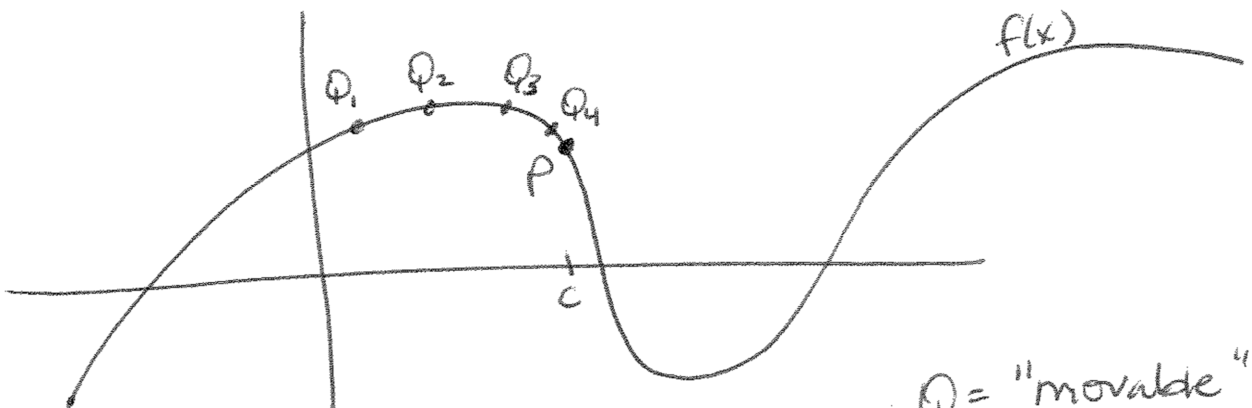
2.1 Two Problems w/ One Theme

Archimedes – slope of a tangent line

Kepler/Galileo/Newton – instantaneous velocity



Q = "movable" pt

P = pt in question

<u>secant line</u> ⇒ line thru P + Q.

<u>tangent line</u> ⇒ limiting positn (if it exists) of secant line as Q moves twd P along the curve.

slope of secant line

$$m = \frac{f(c+h) - f(c)}{\cancel{c}+h-c} = \frac{f(c+h)-f(c)}{h}$$

slope of tangent line

$$m = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$

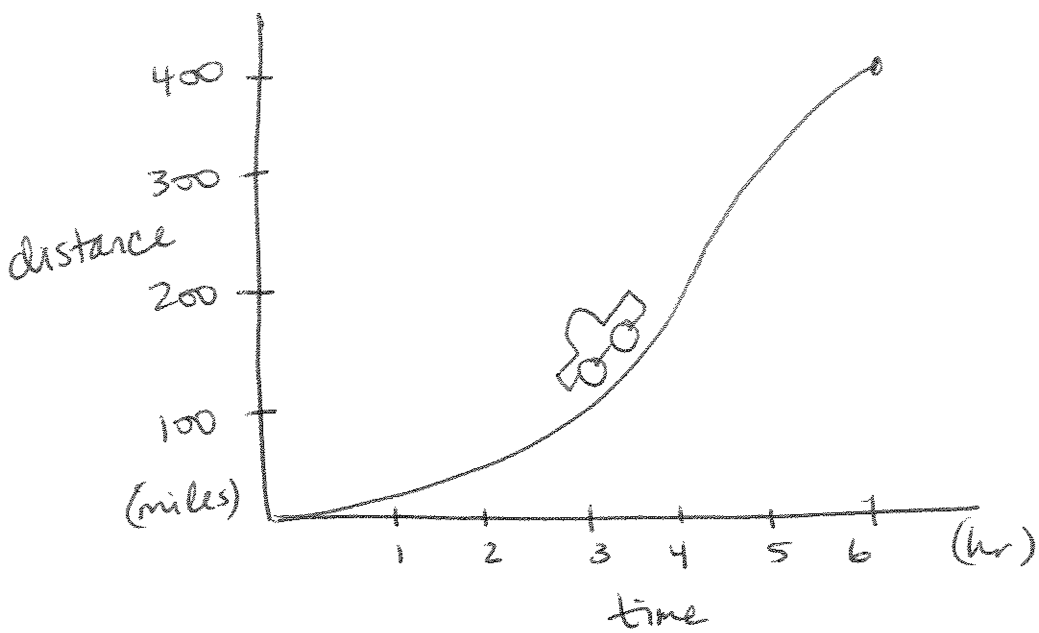
math1210

2.1 (continued)

Ex 1 Find eqn of tangent line to $y = 2/x$ at $x = 1$.

Ex 2 Find slope of $y = -x^2 + 3x$ when $x = -1, 2 \& 5$.

2.1 (continued)



t	d
3	120
2.5	75
2.1	54
2	50

If it takes me 6 hrs to drive 400 miles, then my avg. velocity is $\frac{400}{6} \approx 67$ mi/hr.

But, surely I didn't drive that speed the whole time.

$V_{avg} = ?$ for different time intervals

start t	end t	V_{avg}
2	3	
2	2.5	
2	2.1	

$$V_{avg} = \frac{d_{end} - d_{start}}{t_{end} - t_{start}}$$

$\Rightarrow$ velocity at time $t = 2$ hrs $=$

2.1 (continued)

⇒ Geometrically finding slope of tangent line to a curve is exactly the same mathematical calculation as finding the instantaneous velocity for a moving object.

Ex 3 (#14) An object travels along a line so that its positn s is given by $s(t) = t^2 + 1$ (measured in meters, t measured in seconds).

(a) What is its avg velocity on interval $2 \le t \le 3$?

(b) Avg velocity on $2 \le t \le 2.003$?

(c) avg velocity on $2 \le t \le 2+h$?

(d) Instantaneous velocity at $t = 2$?

★ "rate of change" means instantaneous rate of change

2.2 The Derivative

Defn Derivative

The derivative of f is another function f'

$$\ni \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \forall x$$

provided the limit exists and is finite, for some x-value.

If $f'(c)$ exists, we say $f(x)$ is differentiable at $x = c$.

Ex 1 Find $f'(x)$ given $f(x) = 2\sqrt{x-1}$, $x \geq 1$

2.2 (continued)

Another form of defn of derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Ex 2 Use above defn of f' to find h'(c) if $h(x) = \frac{3}{x-5}$.

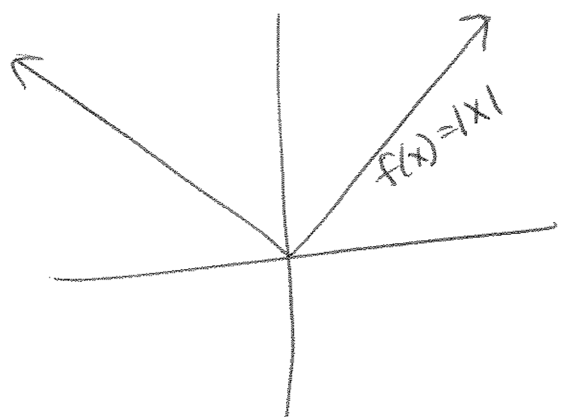
2.2 (continued)

Ex 3 Each of these is a derivative for some function. Can you find the function?

(a) $$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 - 2(x+h) - (3x^2 - 2x)}{h}$$

(b) $$\lim_{x \to 3} \frac{\frac{4}{x} - \frac{4}{3}}{x-3}$$

Ex 4 Let $f(x) = |x|$



Try to find $f'(0)$:

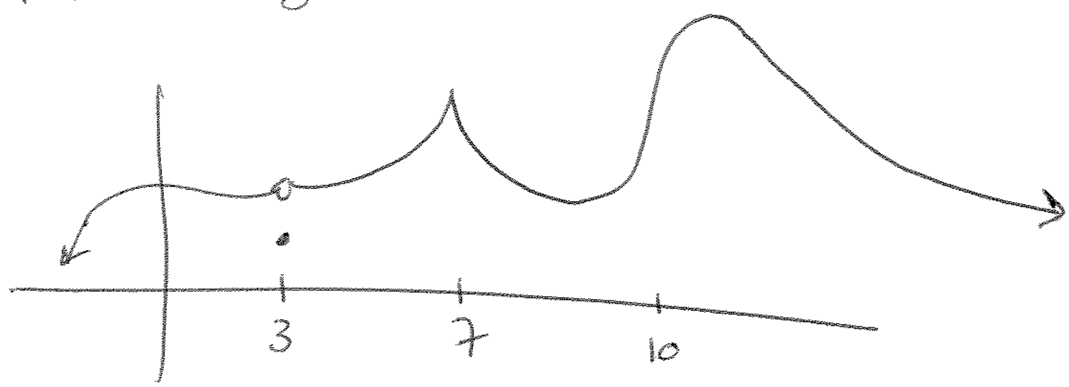
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} = \begin{cases} 1 & \text{if } h \to 0^+ \\ -1 & \text{if } h \to 0^- \end{cases}$$

$\Rightarrow$ $f'(0)$ DNE

2.2 (continued)

Visually, we can see a pt where the derivative (slope) DNE by looking for "corners" or vertical tangents in the graph of the function.



What can we say about derivative of this function at $x=3, 7$ and 10?

Thm <u>Differentiability $\Rightarrow$ Continuity</u>
If $f'(c)$ exists, then f is continuous at $x=c$.

Also if $f(c)$ is discontinuous $x=c$, then $f'(c)$ DNE.

2.3 Rules For Finding Derivatives

Thm **Constant Fn Rule**

If $f(x)=k$, $k \in \mathbb{R}$, $f'(x)=0$ (or $D_x(k)=0$).

Identity Fn Rule

If $f(x)=x$, then $f'(x)=1$ (or $D_x(x)=1$).

Power Rule

If $f(x)=x^n$, $n \in \mathbb{Z}^+$, $f'(x)=nx^{n-1}$ (or $D_x(x^n)=nx^{n-1}$).

D_x is linear operator

Constant Multiple Rule

If $k \in \mathbb{R}$, $f'(x)$ exists, then $D_x[kf(x)] = k(D_x(f(x)))$.

Sum & Difference Rule

If f' & g' exist, then

$$D_x[f(x) \pm g(x)] = D_x f(x) \pm D_x g(x)$$

Ex 1 Find $f'(x)$ if $f(x)=3x^7-4x^6+x^5+2x^3-x^2+4$

2.3 (continued)

Product Rule

If f + g are differentiable, then

$$D_x[f(x)g(x)] = f(x)\,D_x[g(x)] + D_x[f(x)]\,g(x)$$

i.e. $(fg)' = f'g + g'f = g'f + f'g$

Ex 2 Find $f'(x)$ for $f(x) = (2x^3 - 4x + 1)(3x + 5)$

Use product rule:

Multiply out + use power rule to check:

2.3 (continued)

Quotient Rule

Let f + g be differentiable functions, $g(x) \neq 0$.

Then $D_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\, D_x[f(x)] - f(x)\, D_x[g(x)]}{g^2(x)}$

i.e. $\left(\frac{f}{g}\right)'(x) = \frac{g(x) f'(x) - f(x) g'(x)}{g^2(x)}$

☆ I usually remember this one a bit differently.

If $f = \frac{\text{high}}{\text{low}}$, then $f' = \frac{\text{low} \cdot d(\text{high}) - \text{high} \cdot d(\text{low})}{\text{low}^2}$

"low d hi minus hi d lo over lo squared"

Ex 3 Find $f'(x)$ if $f(x) = \frac{2x^2+4x-1}{3x-2}$

2.3 (continued)

Ex 4 $y = \frac{-3}{x} + \frac{2}{x^4 - 7x}$ Find y'!

$\Rightarrow \boxed{D_x(x^{-n}) = -nx^{-n-1}}$ i.e. the Power Rule is true for -ve integers, too!

because

$f(x) = x^{-n} = \frac{1}{x^n} \Rightarrow f'(x) = \frac{x^n(0) - 1(nx^{n-1})}{x^{2n}}$

$= \frac{-nx^{n-1}}{x^{2n}} = -nx^{n-1-2n}$

$= -nx^{-n-1}$

2.3 (continued)

Ex 5 Find $f'(x)$ if $f(x) = \frac{5x-4}{3x^2+1}$

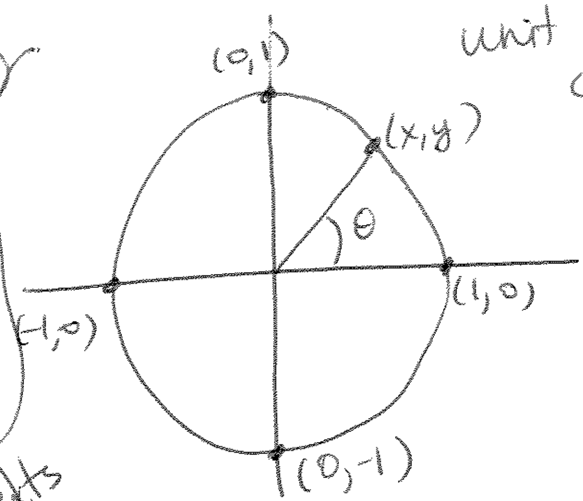
Ex 6 Find y' if $y = 3x(x^3-2x+1)$

Ex 7 Find $g'(x)$ if $g(x) = \frac{-3}{x^5} + \frac{2}{x}$

0.7 Trigonometric Fns

①

unit circle $(r=1)$
centered at $(0,0)$



$\cos\theta = x$

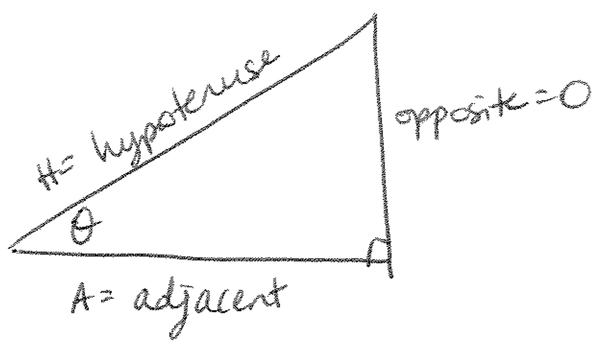
$\sin\theta = y$

$-1 \le \sin\theta \le 1$

$-1 \le \cos\theta \le 1$

two contexts

②



$\sin\theta = \dfrac{O}{H}$

$\cos\theta = \dfrac{A}{H}$

From Pythagorean Thm, we know

$A^2 + O^2 = H^2$ (for above triangle)

$\Leftrightarrow \dfrac{A^2}{H^2} + \dfrac{O^2}{H^2} = 1$ (divide both sides by H^2)

$\Leftrightarrow \left(\dfrac{A}{H}\right)^2 + \left(\dfrac{O}{H}\right)^2 = 1$

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

or we write it this way $\boxed{\cos^2\theta + \sin^2\theta = 1}$

0.7 (continued)

Other Trig Fns

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\qquad$ $\sec\theta = \dfrac{1}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ $\qquad$ $\csc\theta = \dfrac{1}{\sin\theta}$

If we look at $\sin^2\theta + \cos^2\theta = 1$, then

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \iff \boxed{1 + \cot^2\theta = \csc^2\theta}$$

Also $$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \iff \boxed{\tan^2\theta + 1 = \sec^2\theta}$$

Other Trig Properties

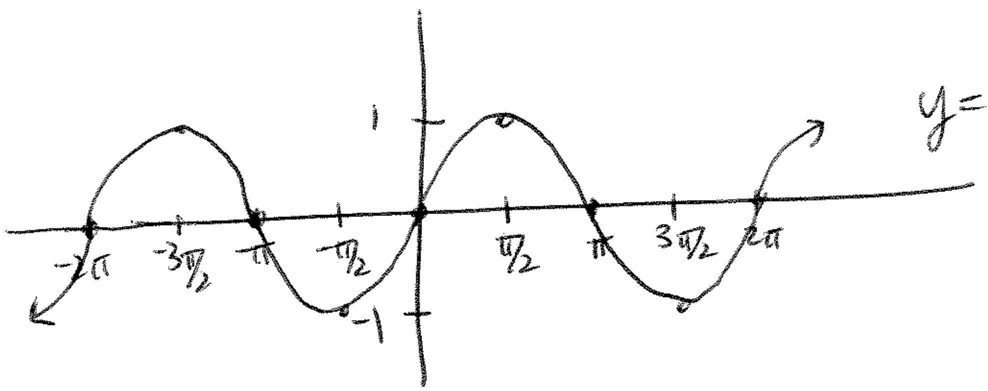
① $\sin\theta = \sin(\theta + 2n\pi)$ $\quad \forall n \in \mathbb{Z}$
& $\cos\theta = \cos(\theta + 2n\pi)$ $\quad \forall n \in \mathbb{Z}$

② $\sin\theta$ is an odd fn, i.e. $-\sin\theta = \sin(-\theta)$
& $\cos\theta$ is an even fn, i.e. $\cos\theta = \cos(-\theta)$
(You can see this is true on the unit circle.)

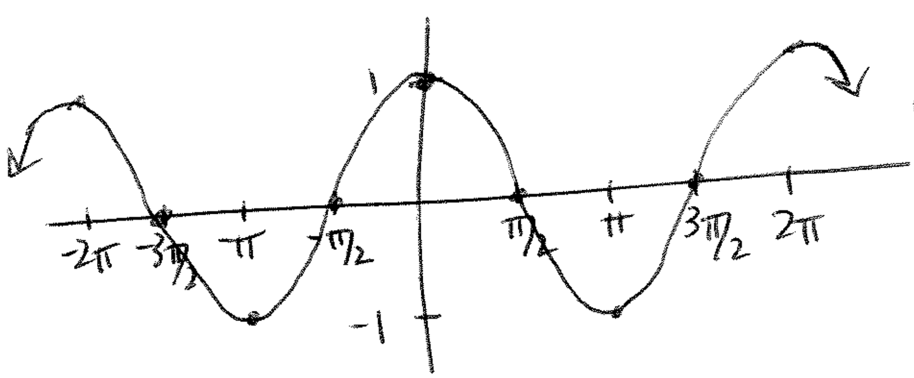
③ $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ &
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

0.7 (continued)

Graphs of $y = \sin x$ & $y = \cos x$



$y = \sin x$



$y = \cos x$

same shape – one graph is just shifted by $\pi/2$ horiz.

<u>amplitude</u> ⇒ half the distance between lowest to highest height of graph

<u>period</u> ⇒ the smallest # p ∋ $f(x+p) = f(x)$ for some fn $f(x)$.

For $\sin x$ & $\cos x$, the period is 2π, i.e. every 2π interval (on x-axis), the curve repeats itself.

0.7 (continued)

$f(x) = a\sin(b(x+c)) + d$ (likewise for $g(x) = a\cos(b(x+c)) + d$)

amplitude

period of curve is $\frac{2\pi}{b}$

horizontal shift c units

vertical shift d units

$\boxed{180° = \pi \text{ (radians)}}$

(★ Note Trig Properties in nice box on pg 47!)

Ex 1 (a) Convert $\frac{-\pi}{3}$ to degrees.

(b) Convert $\frac{3\pi}{18}$ to degrees.

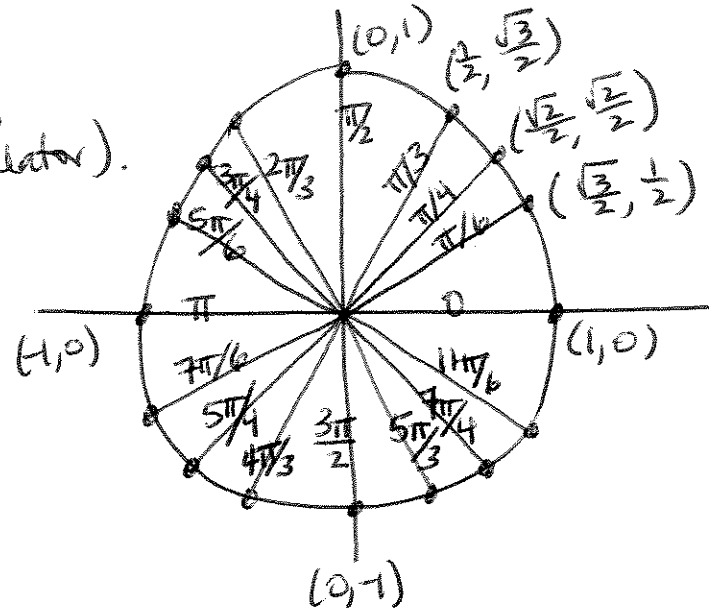
(c) Convert $-120°$ to radians.

(d) Convert $600°$ to radians.

0.7 (continued)

Ex 2

Evaluate (w/o calculator).



(a) $\tan(\pi/6)$

(b) $\sec(\pi/3)$

(c) $\csc(\pi/4)$

(d) $\cos(-\pi/3)$

(e) $\cot(5\pi/6)$

(f) $\sin(5\pi/4)$

0.7 (continued)

Ex 3 For the following fns, list the amplitude, period, horizontal + vertical shift + then graph.

(a) $y = 3\cos(x - \pi/2) - 1$

(b) $y = 2\sin(x + \pi/4)$

(c) $y = \frac{1}{2}\cos(2(x + \pi/2)) + 3$

0.7 (continued)

Ex 4 Verify the identity

$$\cos(3x) = 4\cos^3 x - 3\cos x$$

0.7 (continued)

Ex 5 (# 30)

$\cos^2(\pi/12) = ?$

We know $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$

$\Rightarrow \cos^2\left(\frac{x}{2}\right) = \frac{1+\cos x}{2}$

$\cos^2(\pi/12) = \cos^2\left(\frac{\pi/6}{2}\right)$

$=$

1.4 Limits Involving Trig Fns

Thm $\forall c \in \mathbb{R}$ in the function's domain,

$\lim_{x \to c} \sin x = \sin c$ $\qquad$ $\lim_{x \to c} \csc x = \csc c$

$\lim_{x \to c} \cos x = \cos c$ $\qquad$ $\lim_{x \to c} \sec x = \sec c$

$\lim_{x \to c} \tan x = \tan c$ $\qquad$ $\lim_{x \to c} \cot x = \cot c$

i.e. Basically, we can still just plug in $x=c$, if it works.

Special Trig limits Thm

① $\lim_{t \to 0} \frac{\sin t}{t} = 1$ & ② $\lim_{t \to 0} \frac{1-\cos t}{t} = 0$

pf (of ②)

$$\lim_{t \to 0} \frac{1-\cos t}{t} = \lim_{t \to 0} \frac{(1-\cos t)}{t}\left(\frac{1+\cos t}{1+\cos t}\right)$$

$$= \lim_{t \to 0} \frac{1-\cos^2 t}{t(1+\cos t)} = \lim_{t \to 0} \frac{\sin^2 t}{t(1+\cos t)}$$

$$= \left[\lim_{t \to 0} \frac{\sin t}{t}\right]\left[\lim_{t \to 0} \frac{\sin t}{1+\cos t}\right]$$

$$= 1 \cdot 0 = 0$$

//

1.4 (continued)

Ex 1 $\lim_{x \to 0} \frac{3x \tan x}{\sin x}$

Ex 2 $\lim_{t \to 0} \frac{\sin^2 3t}{2t}$

1.4 (continued)

Ex 3 $\displaystyle\lim_{\theta \to 0} \frac{\tan(5\theta)}{\sin(2\theta)}$

(Hint: $\sin 5\theta = \sin(3\theta + 2\theta) = \sin 3\theta \cos 2\theta + \cos 3\theta \sin 2\theta$

and $\sin 3\theta = \sin(\theta + 2\theta) = \sin\theta \cos 2\theta + \cos\theta \sin 2\theta$)

$$\lim_{\theta \to 0} \frac{\tan 5\theta}{\sin 2\theta} = \lim_{\theta \to 0} \frac{\sin 5\theta}{\sin 2\theta}\left(\frac{1}{\cos 5\theta}\right)$$

$=$