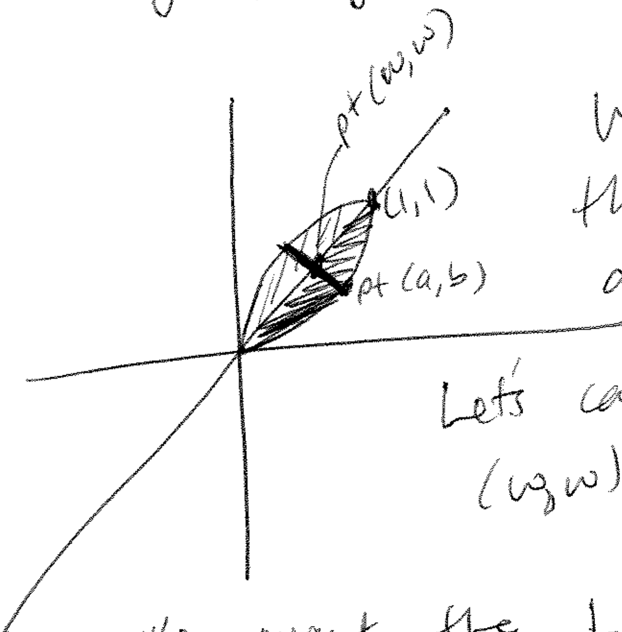
$\boxed{5.3 \ \#23\ (c)}$ find volume

$y = x, \quad y = x^2$ about line $y = x$



We basically want to do the disk method, w/ disks $\perp$ to axis of rotation.

Let's call a random pt on $y=x$ (w,w) (where $w =$ x-value) $\Rightarrow$
$0 \le w \le 1$

We want the $\perp$ line to $y=x$ that goes thru (w,w).

$\Rightarrow \quad m = -1, \quad (w,w)$

$y - w = -(x - w)$

$y = -x + 2w$ ($\perp$ line to $y=x$ thru (w,w))

Basically, our volume will be

$V = \pi \int r^2 \, dw$ where w is given above

and $r =$ distance from (w,w) to (a,b)

(as in picture)

What is (a,b)? It is pt of intersection between $y=-x+2w$ and $y=x^2$. Use substitution (to solve for x in terms of w).

$$-x+2w=x^2$$

$$x^2+x-2w=0 \Rightarrow x=\frac{-1\pm\sqrt{1-4(1)(-2w)}}{2(1)}$$

$$x=\frac{-1\pm\sqrt{1+8w}}{2}$$

but we want x to be positive (in Quad 1)

$$\Rightarrow x=\frac{-1+\sqrt{1+8w}}{2} \qquad \text{i.e. } \boxed{a=\frac{-1+\sqrt{1+8w}}{2}}$$

To find b (y-coord of pt), plug in a to $y=-x+2w$ (for x).

$$y=-\left(\frac{-1+\sqrt{1+8w}}{2}\right)+2w$$

$$y=\frac{1-\sqrt{1+8w}}{2}+\frac{4w}{2}=\frac{1+4w-\sqrt{1+8w}}{2}$$

$$\Rightarrow \boxed{b=\frac{1+4w-\sqrt{1+8w}}{2}}$$

So, r = distance from (w,w) to (a,b)

i.e. $r = \sqrt{(w-a)^2 + (w-b)^2}$

$$\Rightarrow r^2 = (w-a)^2 + (w-b)^2$$

$$\Rightarrow r^2 = \left(w - \left(\frac{-1+\sqrt{1+8w}}{2}\right)\right)^2 + \left(w - \left(\frac{1+4w-\sqrt{1+8w}}{2}\right)\right)^2$$

$$= w^2 - w(-1+\sqrt{1+8w}) + \left(\frac{-1+\sqrt{1+8w}}{2}\right)^2$$
$$+ w^2 - w(1+4w-\sqrt{1+8w}) + \left(\frac{1+4w-\sqrt{1+8w}}{2}\right)^2$$

$$= w^2 + w - w\sqrt{1+8w} + \frac{1}{4} - \frac{1}{2}\sqrt{1+8w} + \frac{1}{4}(1+8w)$$
$$+ w^2 - w - 4w^2 + w\sqrt{1+8w} + \frac{1}{4} + w - \frac{1}{4}\sqrt{1+8w} + w + 4w^2$$
$$- w\sqrt{1+8w} - \frac{1}{4}\sqrt{1+8w} - w\sqrt{1+8w} + \frac{1}{4}(1+8w)$$

$$r^2 = 2w^2 + 6w + 1 - \sqrt{1+8w} - 2w\sqrt{1+8w}$$

$$\Rightarrow V = \pi \int_0^1 r^2 \, dw$$

$$= \pi \int_0^1 2w^2 + 6w + 1 - \sqrt{1+8w} - 2w\sqrt{1+8w} \, dw$$

$$= \pi \left(\frac{2}{3}w^3 + 3w^2 + w\right)\Big|_0^1 - \pi \int_0^1 \sqrt{1+8w} + 2w\sqrt{1+8w} \, dw$$

Let $u = 1 + 8w$

$du = 8\,dw$

$$V = \pi\left(\frac{2}{3}w^3 + 3w^2 + w\right)\Big|_0^1 - \pi\int_0^1 (1+8w)^{1/2} + 2w(1+8w)^{1/2}\, dw$$

$u = 1+8w \Leftrightarrow w = \frac{u-1}{8}$

$du = 8\,dw$

$\frac{1}{8}du = dw$

$w=0,\ u=1$
$w=1,\ u=9$

$$V = \pi\left[\left(\frac{2}{3}+3+1\right)-0\right] - \pi\int_1^9 \left[u^{1/2} + 2\left(\frac{u-1}{8}\right)u^{1/2}\right]\left(\frac{1}{8}\right) du$$

$$= \frac{14}{3}\pi - \frac{\pi}{8}\int_1^9 \left[u^{1/2} + \left(\frac{1}{4}u - \frac{1}{4}\right)u^{1/2}\right] du$$

$$= \frac{14}{3}\pi - \frac{\pi}{8}\int_1^9 \left[u^{1/2} + \frac{1}{4}u^{3/2} - \frac{1}{4}u^{1/2}\right] du$$

$$= \frac{14}{3}\pi - \frac{\pi}{8}\int_1^9 \frac{3}{4}u^{1/2} + \frac{1}{4}u^{3/2}\; du$$

$$= \frac{14}{3}\pi - \frac{\pi}{8}\left(\frac{3}{4}\left(\frac{2}{3}u^{3/2}\right) + \frac{1}{4}\left(\frac{2}{5}u^{5/2}\right)\right)\Big|_1^9$$

$$= \frac{14}{3}\pi - \frac{\pi}{8}\left(\frac{1}{2}u^{3/2} + \frac{1}{10}u^{5/2}\right)\Big|_1^9 = \frac{14}{3}\pi - \frac{\pi}{8}\left[\left(\frac{27}{2}+\frac{243}{10}\right)-\left(\frac{1}{2}+\frac{1}{10}\right)\right]$$

$$= \frac{14}{3}\pi - \frac{\pi}{8}\left[\frac{378}{10} - \frac{6}{10}\right] = \frac{14}{3}\pi - \frac{\pi}{8}\left[\frac{186}{5}\right]$$

$$= \frac{14}{3}\pi - \frac{93\pi}{20} = \boxed{\frac{\pi}{60}}$$