6.3 Future Value of Annuities

annuity $\Rightarrow$ financial plan characterized by regular payments

ordinary annuity $\Rightarrow$ payments made at end of each equal payment interval

annuity due $\Rightarrow$ payments made at beginning of each equal payment interval

Ex 1 Suppose you invest \$1000 at the end of each year for 5 years in an account that pays 10% interest compounded annually. What is the value after 5 years?

end of yr 1: contribute \$1000 and it compounds for 4 yrs $\Rightarrow$ $S_1 = 1000(1+0.1)^4$

end of yr 2:

Generally, then, for an ordinary annuity, the future value is

$$S = \frac{R\left(1-(1+i)^n\right)}{1-(1+i)}$$

$$\Longleftrightarrow\quad S = \frac{R\left(1-(1+i)^n\right)}{-i}$$

$$\Longleftrightarrow\quad \boxed{S = R\left(\frac{(1+i)^n-1}{i}\right)}$$

Future value of ordinary annuity

where R = amt deposited at end of each period

n = # of periods (payments)

$\quad$ = (# of compounding periods per year) * (# yrs) = mt

$i = \frac{r}{m}$ $\quad$ m = # compoundings per year

Twins Story $\Rightarrow$ Two twins invest differently!

Twin 1: At end of college, she invests \$2000 at the end of each year for 8 years in an account that earns 10%, compounded annually. After 8 years, nothing is contributed, but it earns 10% compounded annually for 36 more years. How much does she have at age 65?

6.3 (cont)

Twin 2: At end of college, he invests nothing ~~for~~
8 years. Then he puts in $2000 at the end of
each year for 36 years in an account paying
10% interest compounded annually. How much
does he have at age 65?

6.3 (cont)

Ex 2 How much will be invested at the end of each year at 12% compounded quarterly to pay off a debt of $30,000 in 6 years?

Sinking Fund

When borrowers make periodic deposits that will produce a particular sum on a specific date (another example of ordinary annuity when deposits are regular)

$$R = S\left(\frac{i}{(1+i)^n - 1}\right)$$

Annuities Due

$$S = R\left[\frac{(1+i)^{n+1}-1}{i}\right] - R = R\left[\frac{(1+i)^{n+1}-1}{i} - 1\right]$$

$$= R\left[\frac{(1+i)^{n+1}-1-i}{i}\right] = R\left[\frac{(1+i)^{n+1}-(1+i)}{i}\right]$$

6.3 (cont)

$$\boxed{S_{due} = R(1+i)\left[\frac{(1+i)^n - 1}{i}\right]}$$

Future Value of an Annuity Due

Ex 3 Find the future value of an account with \$100 deposited at the beginning of each month for 5 years into an account that pays 8% compounded quarterly.

6.4 Present Value of Annuities

Present value of an annuity $\Rightarrow$ when we leave a lump sum of \$ in an account and make regular withdrawals

Ordinary annuity $\rightarrow$ w/d at end of each period
annuity due $\rightarrow$ w/d at beginning of each period

Ex 1 You want to withdraw \$1000 at the end of each year from an account that earns 10% interest compounded annually for 4 yrs. How much needs to be in the account from the start?

From compounding interest formula

$S = P(1+\frac{r}{m})^{mt}$ let $i = \frac{r}{m}$ $n = mt$

$\Rightarrow S = P(1+i)^n \Leftrightarrow P = S(1+i)^{-n}$

after 1st yr: we need $P_1 = 1000(1+0.1)^{-1} = 1000(1.1)^{-1}$

after 2nd yr: we need $P_2 = 1000(1.1)^{-2}$

after 3rd yr:

after 4th yr:

$P_1 + P_2 + P_3 + P_4 =$

6.4 (cont)

This leads into general formula for present value of an annuity.

For our example, we had

$$A_n = \frac{1000\left(1-(1.1)^{-(n+1)}\right)}{1-(1.1)^{-1}} - R$$

in general $\Rightarrow$

$$A_n = \frac{R\left(1-(1+i)^{-(n+1)}\right)}{1-(1+i)^{-1}} - R$$

$$= \frac{R\left(1-(1+i)^{-(n+1)}\right)}{1-\frac{1}{1+i}} - R \qquad = \frac{R\left(1-(1+i)^{-(n+1)}\right)}{\frac{1+i-1}{1+i}} - R$$

$$= \frac{R\left(1-(1+i)^{-(n+1)}\right)}{\frac{i}{1+i}} - R \qquad = \frac{R(1+i)\left(1-(1+i)^{-(n+1)}\right)}{i} - \frac{Ri}{i}$$

$$= \frac{R\left(1+i-(1+i)^{-n}-i\right)}{i} = \boxed{\frac{R\left(1-(1+i)^{-n}\right)}{i} = A_n}$$

(Present value of ordinary annuity)

6.4 (cont)

Ex 2 Find the present value of annuity that pays $4000 at the end of each month from an account that earns 8% interest compounded monthly for 25 years.

Ex 3 An inheritance of $400,000 will provide how much at the end of each year for the next 20 years, if money is worth 7%, compounded annually?

6.4 (cont)

annuity due

$$A_n = R + A_{n-1} = R + R\left(\frac{1-(1+i)^{-(n-1)}}{i}\right)$$

an. due ↑ ord. an ↑

$$A_n = \frac{Ri}{i} + \frac{R(1-(1+i)^{-(n-1)})}{i}$$

$$= \frac{R((1+i)-(1+i)^{-(n-1)})}{i}$$

Present value of Annuity Due

$$\boxed{A_{n,due} = \frac{R(1+i)(1-(1+i)^{-n})}{i} = A_n(1+i)}$$

Ex 4 A lottery prize worth \$1,800,000 is awarded in payments of \$10,000 at the beginning of each month for 15 years. Suppose money is worth 7% compounded monthly. What is the real value of the prize?

6.4 (cont)

Deferred Annuity $\Rightarrow$ where 1st payment is deferred until a later date at which pt regular payments are made

For k periods, it just sits there earning interest (compounded).

$\Rightarrow A_n$ grows to $A_n(1+i)^k$

Then $A_n(1+i)^k$ becomes present value of ordinary annuity for n periods.

$$\boxed{A_{(n,k)} = R\left[\frac{1-(1+i)^{-n}}{i(1+i)^k}\right]}$$ Present value of Deferred Annuity deferred for k periods & pays n periods

Ex 5 Carol received a trust fund inheritance of $10000 on her 30th birthday. She plans to use it to supplement her income w/ 20 qrtrly payments beginning on her 60th birthday. If money is worth 7.6% compounded quarterly, how much will each payment be?

6.5 Loans and Amortization

<u>amortization</u> $\Rightarrow$ when loan is repayed by making all payments equal (i.e installment loan)

Bank is basically investing a lump sum of \$ and getting a periodic return which is exactly like present value of ordinary annuity!

$$\Rightarrow \quad A_n = R\left[\frac{1-(1+i)^{-n}}{i}\right] \iff A_n i = R\left[1-(1+i)^{-n}\right]$$

$$\iff \boxed{R = A_n\left[\frac{i}{1-(1+i)^{-n}}\right]}$$

Amortization Formula

<u>Ex 1</u> When you graduate college, you buy a new car and can afford a monthly payment of \$250/mo. If you get a special rate of 3.5% interest, compounded monthly, for 5 years, how much can you afford to borrow?

6.5 (cont)

Ex 2 Angie buys a house for \$200,000. She puts \$25,000 down and she gets a loan for the rest at 5.5% compounded monthly for 20 yrs. What will her payments be?

Amortization Schedule

A loan of \$10,000 w/ interest rate of 10% could be repaid in 5 equal annual payments.

$$R = 10000\left[\frac{0.1}{1-(1.1)^{-5}}\right] \doteq \$2,637.97$$

each \$2637.97 is used to pay some principal + some interest

	payment	interest	principal	unpaid balance
1	2637.97	0.1(10,000) = 1000	1637.97	8362.03
2	2637.97	0.1(8362.03) = 836.20	1801.77	6560.26
3	2637.97	0.1(6560.26) = 656.03	1981.94	4578.32
4	2637.97	0.1(4578.32) = 457.83	2180.14	2398.18
5	2637.97	0.1(2398.18) = 239.8	2398.18	0

6.5 (cont)

Unpaid balance of (payoff) loan = present value needed to generate all remaining payments

$$\boxed{A_{n-k} = R\left[\frac{1-(1+i)^{-(n-k)}}{i}\right] \quad \text{unpaid balance of loan (payoff)}}$$

k = # payments made so far
n = # payments for total loan

Ex 3 A company that purchases a piece of equipment by borrowing \$250,000 for 10 years at 6%, compounded monthly, has monthly payments of \$2775.51.

(a) Find the unpaid balance on loan after 1 year.

(b) During that first year, how much interest does the company pay?