6.3 Future Value of Annuities

Annuity = financial plan characterized by regular payments

Ordinary annuity = payments made at end of each equal payment interval

Annuity due = payments made at beginning of each equal payment interval

Ex 1: Suppose you invest $1000 at the end of each year for 5 years in an account that pays 10% interest compounded annually. What is the value after 5 years?

End of yr 1: contribute $1000 and it compounds for 4 yrs \( \Rightarrow S_1 = 1000 (1+0.1)^4 \)

End of yr 2:
Generally, then, for an ordinary annuity, the future value is
\[ S = R \frac{(1 - (1+i)^n)}{1 - (1+i)} \]
\[ S = R \frac{(1 - (1+i)^n)}{-i} \]
\[ S = R \left( \frac{(1+i)^n - 1}{i} \right) \]
Future value of ordinary annuity

where
- \( R \) = amt deposited at end of each period
- \( n \) = \# of periods (payments)
- \( b = (\# \text{ of compounding periods per year}) \times \)
  \( (\# \text{ yrs}) = mt \)
- \( i = \frac{r}{m} \)
- \( m = \# \text{ compoundings per year} \)

**Twins Story**

Two twins invest differently!

**Twins**: At end of college, she invests $2000 at the end of each year for 8 years in an account that earns 10%, compounded annually. After 8 years, nothing is contributed, but it earns 10% compounded annually for 36 more years. How much does she have at age 65?
Twin 2: At end of college, he invests nothing for 8 years. Then he puts in $2000 at the end of each year for 36 years in an account paying 10% interest compounded annually. How much does he have at age 65?
Ex 2 How much will be invested at the end of each year at 12% compounded quarterly to pay off a debt of $30,000 in 6 years?

\[
S = R \left[ \frac{(1+i)^{n+1} - 1}{i} \right] - R = R \left[ \frac{(1+i)^{n+1} - 1}{i} \right] - R
\]

\[
= R \left[ \frac{(1+i)^{n+1} - 1 - i}{i} \right] = R \left[ \frac{(1+i)^{n+1} - (1+i)}{i} \right]
\]

Sinking Fund

When borrowers make periodic deposits that will produce a particular sum on a specific date (another example of ordinary annuity when deposits are regular)

\[
R = S \left( \frac{i}{(1+i)^n - 1} \right)
\]
Ex 3 Find the future value of an account with $100 deposited at the beginning of each month for 5 years into an account that pays 8% compounded quarterly.
6.4 Present Value of Annuities

Present value of an annuity = when we leave a lump sum of $ in an account and make regular withdrawals.

Ordinary annuity = w/d at end of each period
Annuity due = w/d at beginning of each period

Ex1: You want to withdraw $1000 at the end of each year from an account that earns 10% interest compounded annually for 4 yrs. How much needs to be in the account from the start?

From compounding interest formula
\[ S = P \left(1 + \frac{i}{m}\right)^{mt} \]
\[ i = \frac{r}{m}, \quad n = mt \]

\[ S = P \left(1 + i\right)^n \Rightarrow P = S \left(1 + i\right)^{-n} \]

After 1st yr: we need \( P_1 = 1000 \left(1 + 0.1\right)^{-1} = 1000 \left(1.1\right)^{-1} \)

After 2nd yr: we need \( P_2 = 1000 \left(1.1\right)^{-2} \)

After 3rd yr:

After 4th yr:

\[ P_1 + P_2 + P_3 + P_4 = \]
6.4 (cont)

This leads into general formula for present value of an annuity.

For our example, we had

\[ A_n = \frac{1000(1-(1+i)^{-1})}{1-(1+i)^{-1}} - R \]

in general

\[ A_n = \frac{R(1-(1+i)^{-(n+1)})}{1-(1+i)^{-1}} - R \]

\[ = \frac{R(1-(1+i)^{-(n+1)})}{1-i} - R \]

\[ = \frac{R(1-(1+i)^{-(n+1)})}{i} \cdot \frac{i}{1+i} - R \]

\[ = \frac{R(i^{n+1} - (1+i)^n)}{i} \]

\[ = \frac{R(1-(1+i)^{-n})}{i} = A_n \]

Present Value of Ordinary Annuity
6.4 (cont)

Ex 2 Find the present value of annuity that pays $4000 at the end of each month from an account that earns 8% interest compounded monthly for 25 years.

Ex 3 An inheritance of $400,000 will provide how much at the end of each year for the next 20 years, if money is worth 7%, compounded annually?
6.4 (cont)

annuity due

\[ A_n = R + A_{n-1} = R + R \left( \frac{1 - (1+i)^{-(m+1)}}{i} \right) \]

\[ A_n = \frac{R}{i} + \frac{R}{i} \left( 1 - (1+i)^{-(m+1)} \right) \]

\[ = R \frac{(1+i) - (1+i)^{-(m+1)}}{i} \]

Present value of Annuity Due

\[ A_{n, \text{due}} = \frac{R(1+i)(1-(1+i)^{-n})}{i} = A_n (1+i) \]

Ex. 9 A lottery prize worth $1,800,000 is awarded in payments of $10,000 at the beginning of each month for 15 years. Suppose money is worth 7% compounded monthly. What is the real value of the prize?
Deferred Annuity =) where 1st payment is deferred until a later date at which pt regular payments are made.

For k periods, it just sits there earning interest (compounded).

\[ A_n \text{ grows to } A_n(1+i)^k \]

Then \[ A_n(1+i)^k \] becomes present value of ordinary annuity for \( n \) periods.

\[
A_{n+k} = R \left[ \frac{1 - (1+i)^{-n}}{i(1+i)^k} \right] \quad \text{Present Value of Deferred Annuity deferred for } k \text{ periods & pays } n \text{ periods}
\]

Ex 5 Carol received a trust fund inheritance of $10000 on her 30th birthday. She plans to use it to supplement her income of 20 quarterly payments beginning on her 60th birthday. If money is worth 7.6%, compounded quarterly, how much will each payment be?
6.5 Loans and Amortization

Amortization = when loan is repaid by making all payments equal (i.e. installment loan)

Bank is basically investing a lump sum of $ and getting a periodic return which is exactly like present value of ordinary annuity!

\[
A_n = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] \quad \Rightarrow \quad A_n i = R \left[ 1 - (1+i)^{-n} \right]
\]

\[
R = A_n \left[ \frac{i}{1 - (1+i)^{-n}} \right]
\]

Amortization Formula

Example 1: When you graduate college, you buy a new car and can afford a monthly payment of $250/mo. If you get a special rate of 3.5% interest, compounded monthly, for 5 years, how much can you afford to borrow?
Ex 2 Angie buys a house for $200,000. She puts $25,000 down and she gets a loan for the rest at 5.5% compounded monthly for 20 yrs. What will her payments be?

Amortization Schedule
A loan of $10,000 at interest rate of 10% could be repaid in 5 equal annual payments.

\[ R = \frac{10,000 \times 0.1}{1 - (1.1)^{-5}} = \$2,637.97 \]

Each $2,637.97 is used to pay some principal + some interest.
Unpaid balance of loan = present value needed to generate all remaining payments

\[ A_{nt} = R \left[ \frac{1 - (1 + i)^{-m}}{i} \right] \quad \text{unpaid balance (payoff)} \]

\[ k = \# \text{payments made so far} \]
\[ N = \# \text{payments for total loan} \]

Ex 3 A company that purchases a piece of equipment by borrowing $257,000 for 10 years at 6.70%, compounded monthly, has monthly payments of $2775.51.

(a) Find the unpaid balance on loan after 1 year.
(b) During that first year, how much interest does the company pay?