5.1 Area of a Plane Region

\[ A = \{ \text{area under a curve } f(x) \text{, from } x = a \text{ to } x = b \} \]

\[ \int_a^b f(x) \, dx \]

**Ex 1** Find area of region under \( f(x) = x^3 - x + 2 \) (bounded below by x-axis), on \([-1, 2]\).

\[ A = \int_{-1}^{2} x^3 - x + 2 \, dx \]

**Process**

1. Sketch graph.
2. Slice into thin pieces and label. (decide dx or dy)
3. Decide on integration bounds.
4. Take integral of function.
Ex 2  Find area between $y = \sqrt{x} - 10$ and $y=0$, between $x=0$ and $x=9$. 
5.1 (continued)

Ex 3  Find area between $y = x^2 - 9$ and $y = (2x-1)(x+3)$. 
Ex 4 Find the area of the region bounded by \( x = y^2 - 2y \) and \( x - y - 4 = 0 \).
5.2 Volumes of Solids (Slabs, Disks, Washers)

Definite integral = \( \int \) sum of thin slices of something where the slice has a thickness of \( dx \) (or \( dy \))

Volume of right prism/cylinder =

\[
V = Ah
\]

where \( A = \text{area of base} \)

\[
V = Ah
\]

\[
V = Ah
\]

\[
V = Ah
\]

\[
V = A_\text{penny} \Delta x \quad \text{where} \quad \Delta x = \text{thickness of penny}
\]

To find volume of a stack of pennies,

\[
V = \sum_{i=1}^{n} A \Delta x
\]

(looks like a Riemann sum)

Imagine we have a 3d solid like this & we want to find the volume with a thickness \( dx \).

We can take a "slice" off it.

The very thin slice will look like

\[
\Rightarrow \quad V = \sum_{i=1}^{n} \frac{1}{2} A(x_i) \Delta x
\]

\[
\Rightarrow \quad V = \lim_{n \to \infty} \frac{1}{2} A(x_i) \Delta x
\]

\[
\Rightarrow \quad V = \int_{a}^{b} A(x) \, dx
\]

where \( A(x) \) is the area of the circular slice.
5.2 (continued)

We will now find the volume of a solid of revolution; i.e. a 3d solid generated by revolving a 2d curve about an axis in a 2d plane.

Ex 1 Find the volume of the solid of revolution obtained by revolving the region bounded by $y = \sqrt{x}$, the x-axis and the line $x = 9$ about the x-axis.

Each slice is a disk with $A = \pi r^2$.

$$V = \int_{0}^{9} \pi r^2 \, dx$$

$$= \pi \int_{0}^{9} (\sqrt{x})^2 \, dx$$

$$= \pi \int_{0}^{9} x \, dx = \frac{\pi}{2} x^2 \bigg|_{0}^{9} = \frac{\pi}{2} (81 - 0)$$

$$= \frac{81\pi}{2}$$
Ex 2. Find the volume of the solid generated by revolving the region enclosed by
\[ x = \frac{2}{y}, \ y = 2, \ y = 6, \ x = 0 \] about the \( y \)-axis.
Ex 3 Find the volume of the solid generated by revolving about the x-axis the region bounded by \( y = 6x + 6x^2 \).

**Washer method**

Each slice here is a washer rather than a circle.

So, Area = \( \pi [R^2 - r^2] \)

Otherwise, this is same as Disk method process.
Ex 4. Find the volume of the solid generated by revolving about the line $y = 2$ the region in the 1st quadrant bounded by the parabolas $3x^2 - 16y + 48 = 0$ and $x^2 - 16y + 80 = 0$ and the y-axis. (Hint: always measure radius from axis of revolution.)
5.3 Volumes of Solids (Shells)

There are 2 methods used to find volume of a solid of revolution:

1. Disk/Washer method
2. Shell method

We know \( V = Ah \) (\( A = \text{area of base} \))

For a shell, then:

\[
V = (\pi r_0^2 - \pi r_i^2)h
= \pi (r_0^2 - r_i^2)h
= \pi (r_0 - r_i)(r_0 + r_i)h
= 2\pi \left( \frac{r_0 + r_i}{2} \right)(r_0 - r_i)h
\]

\( \frac{\Delta r}{2} \) avg. radius

\[
= 2\pi r \Delta rh
\]

If we cut our shell down the side, we get:

\[
V = 2\pi rh \Delta r
\]

Now, we can think of adding up a bunch of "small thickness" shells to get the volume of a solid cylinder. \( V = 2\pi \int_{a}^{b} x f(x) \, dx \)

(See nice picture on pg. 288 of book.)
Ex 1 Find the volume of the solid generated when the region bounded by $y=x^2$, $x=1$, $y=0$ is revolved about the $y$-axis. (Use the shell method.)

Our shell will be like

\[ \text{i.e. the shell thickness is } dx. \]

\[ = V = 2\pi \int_0^1 \text{radius} \cdot \text{height} \, dx \]

radius of shell = $x$ (measured from $y$-axis)

height of shell = $y$ (measured from $x$-axis)

\[ = V = 2\pi \int_0^1 x (x^2) \, dx \]
5.3 (continued)

Ex 2 Find the volume of the solid generated when the region bounded by \( y = 9 - x^2 \) \((x \geq 0)\), \( x=0, y=0 \) is revolved about the y-axis.

1. Shell method:

2. Disk method:
Ex 3 Find the volume of the solid generated when the region bounded by \( y = 9 - x^2 \) (\( x > 0 \)), \( x = 0 \), \( y = 0 \) is revolved about the line \( x = 3 \).
Ex 4. A region $R$ is shown below. Set up an integral for the volume obtained by revolving $R$ about the given line.

(a) The $y$-axis.
(b) The $x$-axis.
(c) The line $y = 3$. 

[Diagram showing region $R$ with boundaries $y = f(y)$ and $x = g(y)$]
5.4 Length of a Plane Curve

A plane curve is a curve that lies in a 2D plane. We can define a plane curve using parametric equations, i.e., by defining \( y = f(t) \) and \( x = g(t) \) as functions of a parameter.

For example, we know from trig that
\[
y = \sin \theta \quad \text{and} \quad x = \cos \theta \quad \text{on unit circle.}
\]
\[
\sin^2 \theta + \cos^2 \theta = 1
\]
\[
\Rightarrow y^2 + x^2 = 1
\]

We know \( x^2 + y^2 = 1 \) is usual eqn for unit circle in a Cartesian coordinate system.

So, we can define a unit circle as

\( 1 \) \( x^2 + y^2 = 1 \)

or

\( 2 \) \( x = \cos \theta \) where \( \theta \) is our "parameter"

\( y = \sin \theta \), \( \theta \in [0, 2\pi] \)

Ex 1: Sketch the graph of the curve given by
\[
x = 3t^2 + 2 \quad \text{and} \quad y = 2t^2 - 1 \quad 1 \leq t \leq 4
\]
5.4 (continued)

When we trace a plane curve given parametrically, we use arrowheads (usually) to indicate orientation of curve, i.e. how it travels as our parameter increases.

**Defn.** A plane curve is **smooth** if it is given by a pair of parametric eqns \( x = f(t), y = g(t), t \in [a,b], \) where \( f' + g' \) exist and are continuous on \([a,b]\), and \( f'(t) + g'(t) \) are not simultaneously zero on \((a,b)\).

**Arc length** (typically \( s = \text{arc length} \))

We can approximate length of a plane curve by adding up lengths of linear segments, between \( Q_i \) (pts on curve).

\[
\Delta w_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}
\]

but \( \Delta x_i = f(t_i) - f(t_{i-1}) \) and \( \Delta y_i = g(t_i) - g(t_{i-1}) \)

\[
\Delta w_i = \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2}
\]

From MVT for derivatives, we know \( \bar{t}_i \) and \( \hat{t}_i \) exist

\[
f(t_i) - f(t_{i-1}) = f'(\bar{t}_i) \Delta t_i \quad \text{w/} \quad \Delta t_i = t_i - t_{i-1}
\]

+ \( g(t_i) - g(t_{i-1}) = g'(\hat{t}_i) \Delta t_i \)
5.4 (continued)

\[ \Delta w_i = \sqrt{(f'(t_i) \Delta t_i)^2 + (g'(t_i) \Delta t_i)^2} \]

\[ = \sqrt{(f'(t_i))^2 + (g'(t_i))^2} \Delta t_i \]

\[ = \sqrt{(f'(t_i))^2 + (g'(t_i))^2} \Delta t_i \]

Approximate length of curve = \[ \sum_{i=1}^{n} \Delta w_i \]

Arc length \[ = \sum_{i=1}^{n} \sqrt{(f'(t_i))^2 + (g'(t_i))^2} \Delta t_i \]

\[ \Rightarrow L = \int_{a}^{b} \sqrt{(f'(t))^2 + (g'(t))^2} \, dt \]

where \( L \) = arc length of plane curve given by \( x = f(t), y = g(t), a \leq t \leq b \)

\[ \Rightarrow L = \int_{a}^{b} \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt \]

\[ \text{If } y = f(x) \text{ (so no parametric eqns), then} \]

\[ L = \int_{a}^{b} \sqrt{1 + (dy/dx)^2} \, dx \]

Likewise, if \( x = g(y) \), then

\[ L = \int_{c}^{d} \sqrt{1 + (dx/dy)^2} \, dy \]
5.4 (continued)

Ex 1 (A classic*) Find the circumference of the circle \(x^2 + y^2 = r^2\).

We can represent this with parametric eqns

\[ x = r \cos \theta \quad y = r \sin \theta \quad \theta \in [0, 2\pi]\]

\[ \Rightarrow L = \int_0^{2\pi} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} \, d\theta \]

\[ = \int_0^{2\pi} \sqrt{r^2 \sin^2 \theta + r^2 \cos^2 \theta} \, d\theta \]

\[ = \int_0^{2\pi} \sqrt{r^2} \, d\theta \]

\[ = \int_0^{2\pi} r \, d\theta \]

\[ = r \int_0^{2\pi} \theta \, d\theta \]

\[ = r \left[ \theta \right]_0^{2\pi} = r (2\pi - 0) = 2\pi r \]

Ex 2 Find length of line segment on \(2y - 2x + 3 = 0\) between \(y = 1\) and \(y = 3\). (Check using distance formula.)
Ex 3 (a) Estimate the arc length of curve $f(x) = \sqrt{x}$ from $x=0$ to $x=4$ by 4 line segments.

\[ L \approx \sum_{i=1}^{4} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \]

but for this problem

$\Delta x_i = 1$ and $\Delta y_i = f(x_i) - f(x_{i-1})$

\[ = \sum_{i=1}^{4} \sqrt{1 + [f(x_i) - f(x_{i-1})]^2} \]

\[ = \sqrt{1 + \left[\sqrt{1} - \sqrt{0}\right]^2} + \sqrt{1 + \left[\sqrt{2} - \sqrt{1}\right]^2} + \sqrt{1 + \left[\sqrt{3} - \sqrt{2}\right]^2} + \sqrt{1 + \left[\sqrt{4} - \sqrt{3}\right]^2} \]

\[ = \sqrt{2} + \sqrt{4 - 2\sqrt{2}} + \sqrt{6 - 2\sqrt{3}} + \sqrt{8 - 2\sqrt{3}} \]

(b) Find arc length now,

\[ L = \int_{a}^{b} \sqrt{1 + \left[f'(x)\right]^2} \, dx = \int_{0}^{4} \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} \, dx \]

\[ = \int_{0}^{4} \sqrt{1 + \frac{1}{4x}} \, dx = \int_{0}^{4} \sqrt{\frac{4x+1}{4x}} \, dx \]

Now what?
5.4 (continued) (Surface Area)

Differential of Arc Length

Let \( f(x) \) be continuously differentiable on \([a,b]\). Start measuring arc length from \((a,f(a))\), up to \((x,f(x))\), where \( a \in \mathbb{R} \).

Then, our arc length is a function of \( x \).

i.e., \( s(x) = \int_{a}^{x} \sqrt{1 + [f'(t)]^2} \, dt \)

\[ \Rightarrow s'(x) = \frac{d}{dx} \left( \int_{a}^{x} \sqrt{1 + [f'(t)]^2} \, dt \right) \]

\[ \frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \]

\[ \Rightarrow ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

or \( ds = \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \)

Surface Area (or Surface of Revolution)

Now, we'll take a plane curve and rotate it about an axis to create a 3d solid. We're interested in its surface area.
5.4 (continued)

Frustum of a cone is a small piece of the cone.

+ we know \( A = 2\pi \left( \frac{r_1 + r_2}{2} \right) l \)

i.e. \( A = 2\pi \) (avg radius of), (slant height) frustum?

We can find surface area by adding up a bunch of little frustum areas!

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi y_i \Delta s_i = \int_{a}^{b} 2\pi y \, ds
= \int_{a}^{b} 2\pi f(x) \, ds
\]

\[
A = \int_{a}^{b} 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx
\]

OR

\[
A = 2\pi \int_{a}^{b} g(t) \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt \quad \text{if parametric eqns}
\]

Ex 1 Find the area of the surface generated by revolving \( y = \sqrt{25 - x^2} \) \( x \in [-2, 3] \) about the x-axis.

\[
A = 2\pi \int_{a}^{b} y \sqrt{1 + (y')^2} \, dx
\]

\[
= 2\pi \int_{a}^{b} y \sqrt{1 + (25 - x^2)'} \, dx
\]}
5.4 (continued)

Ex 2 Find area of the surface generated by revolving
\[ x = 1-t^2, \quad y = 2t, \quad t \in [0,1] \] about the x-axis.
5.5 Work

Work = Force \times Distance \quad (\text{work done by a force})

W = FD

(If force measured in newtons, distance in meters, then work units are joules. If force in lbs and distance in ft, then work is ft-lbs.)

Force is sometimes variable, in which case we need to approximate the work done in little chunks and then add up all the “chunks” of work. And another case for a definite integral. \int_{a}^{b} F(x) \, dx = W

\begin{align*}
W = \lim_{\Delta x \to 0} \sum_{i=1}^{n} F(x_i) \Delta x
\end{align*}

\text{Force at little bit of distance } x_i \text{ \ of distance}

\text{W=work}

\text{F(x)=force functn}

Springs

Hooke’s law says \overline{F(x)=kx} \text{ where } k = \text{spring constant}, \overline{F(x)} = \text{force necessary to keep a spring stretched (or compressed) } x \text{ units beyond (or short of) its natural length.}

Ex 1 (#1) A force of 6 lbs is required to keep a spring stretched \(\frac{1}{2}\) ft beyond its normal length. Find the spring constant. And find the work done in stretching the spring \(\frac{1}{2}\) ft beyond its natural length.
Ex2. A force of 1.8 newtons is required to keep a spring of natural length of 0.5 meter compressed to a length of 0.3 m. Find the work done in compressing the spring from its natural length to a length of 0.2 m.
S.5 (continued)

Ex 3 (#10) A tank of the triangular cross section (as shown) has a length of 10 ft and is full of water. The water is to be pumped to a height of 5 feet above the top of the tank. Find the work done in emptying the tank.

\( F = \text{weight} \) and \( S = 62.4 \text{ lbs/ft}^3 \) is density of water.

(Hint: Work is still \( W = Fd \) and

\[
\begin{align*}
\text{for a triangle, } & F = \frac{1}{2} \times \text{base} \times \text{height} \\
& = \frac{1}{2} \times 3 \times 4 \\
& = 6 \text{ ft-lb}
\end{align*}
\]
5.6 Moments, Center of Mass

This stays balanced only if \( m_1 d_1 = m_2 d_2 \).

If we put a seesaw on the x-axis with fulcrum at origin, then to stay balanced we need to satisfy

\[
x_1 m_1 + x_2 m_2 = 0 \quad \text{(since } x_i = -d_i)\]

Moment of a particle wrt a pt \( = \) product of mass \( m \) of the particle with its directed distance from a pt. (This measures tendency to produce a rotation about that pt.)

Total moment \( M \) for a bunch of masses \( = \sum_{i=1}^{n} x_i m_i \)

Where does fulcrum need to be placed to balance? Let’s call it \( \bar{x} \).
Then, for balance at \( x \), we need

\[(x_1-x)m_1 + (x_2-x)m_2 + \ldots + (x_n-x)m_n = 0\]

\(\Rightarrow\)

\[x_1m_1 + x_2m_2 + \ldots + x_nm_n = \overline{x}m_1 + \overline{x}m_2 + \ldots + \overline{x}m_n\]

\(\Rightarrow\)

\[x_1m_1 + x_2m_2 + \ldots + x_nm_n = \overline{x} \left( m_1 + m_2 + \ldots + m_n \right)\]

\[\overline{x} = \frac{x_1m_1 + x_2m_2 + \ldots + x_nm_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} x_im_i}{\sum_{i=1}^{n} m_i}\]

Balance pt, i.e., center of mass, is just \( M \) (total moment w.r.t. origin) divided by \( n \) (total mass).

For a continuous mass distribution along a line (like in a wire =)

\[\overline{x} = \frac{M}{m} = \frac{\int_{a}^{b} xS(x)\,dx}{\int_{a}^{b} S(x)\,dx} \quad \text{(where \( S(x) \) = density function)}\]

Ex 1: John + Mary, weighing 180 lbs + 110 lbs respectively, sit at opposite ends of a 12-ft teeter totter w/ the fulcrum in the middle, where should their 90-lb son sit in order for the board to balance?
5.6 (continued)

**Ex 2** A straight wire 7 units long has density \( s(x) = 1 + x^3 \) at a point \( x \) units from one end. Find the distance from this end to the center of mass.
Now, consider a discrete set on 2d masses.

Then, to find the center of mass (i.e., the geometric center) \((\bar{x}, \bar{y})\),

we'll have \(\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}\)

where \(M_y = \sum_{i=1}^{n} x_i m_i, \quad M_x = \sum_{i=1}^{n} y_i m_i\)
and \(m = \sum_{i=1}^{n} m_i\)

**Example 3** The masses and coordinates of a system of particles are given by the following: 5, (-3, 2); 6, (2, -2); 2, (3, 5); 7, (4, 3); 1, (7, -1). Find the moments of this system w.r.t. the coord. axes & find center of mass.
5.6 (continued)

Now, consider a continuous 2d region (we'll call it a lamina) that has constant (homogeneous) density everywhere. Then, to find the center of mass \((\bar{x}, \bar{y})\), we'll have (still)

\[
\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}
\]

but

\[
M_y = \delta \int_a^b x[(f(x)-g(x))] \, dx
\]

\[
M_x = \frac{\delta}{2} \int_a^b [f^2(x) - g^2(x)] \, dx
\]

and

\[
m = \delta \int_a^b [f(x)-g(x)] \, dx
\]

because \(m\) used to be \(m = \sum_{i=1}^{N} m_i\), which becomes

\[
m = \sum_{i=1}^{N} \Delta y \cdot \Delta x_i = \sum_{i=1}^{N} \Delta x_i \cdot (f(x_i) - g(x_i))
\]

\[
= \sum_{i=1}^{N} \Delta x_i \cdot f(x_i) - \sum_{i=1}^{N} \Delta x_i \cdot g(x_i)
\]

\[
= \delta \int_a^b x \cdot (f(x)-g(x)) \, dx
\]

and

\[
M_x = \sum_{i=1}^{N} y_i \cdot \Delta x_i = \sum_{i=1}^{N} \left( \frac{f(x_i) + g(x_i)}{2} \right) \left( \delta (f(x_i) - g(x_i)) \Delta x_i \right)
\]

\[
= \frac{\delta}{2} \sum_{i=1}^{N} [f^2(x_i) - g^2(x_i)] \, \Delta x_i
\]

\[
= \frac{\delta}{2} \int_a^b [f^2(x) - g^2(x)] \, dx
\]
\[S.6 \text{ (continued)}\]

\[x = \frac{\bar{M}_y}{m} = \frac{\int_a^b x \,(f(x) - g(x)) \,dx}{\int_a^b [f(x) - g(x)] \,dx}\]

and \[y = \frac{M_x}{m} = \frac{\frac{1}{2} \left( \frac{\int_a^b [f^2(x) - g^2(x)] \,dx}{\int_a^b [f(x) - g(x)] \,dx} \right)}{\int_a^b [f(x) - g(x)] \,dx}\]

Center of mass = Centroid

\[(x, \bar{y})\]

**Ex 4** Find the centroid of the region bounded by

\[y = x^2 \text{ and } y = x + 2\]

**Note:** It doesn't depend at all on density! Only depends on shape = geometric problem.