2.4 Derivatives of Trigonometric Functions

Use the definition of the derivative to find $D_x(\sin x)$.

$$D_x(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cosh h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left[ \sin x \left( \frac{\cosh h - 1}{h} \right) + \cos x \left( \frac{\sinh h}{h} \right) \right]$$

$$= \sin x \left[ \lim_{h \to 0} \left( \frac{\cosh h - 1}{h} \right) \right] + \cos x \left[ \lim_{h \to 0} \frac{\sinh h}{h} \right]$$

$$= -\sin x \left[ \lim_{h \to 0} \frac{1 - \cosh h}{h} \right] + \cos x \left( 1 \right)$$

$$= -\sin x \left( 0 \right) + \cos x$$

$$= \cos x$$

We can follow the same argument to find $D_x(\cos x)$.

$$\implies D_x(\sin x) = \cos x \quad \text{and} \quad D_x(\cos x) = -\sin x$$

Also,

$$D_x(\tan x) = \sec^2 x \quad D_x(\cot x) = -\csc^2 x$$

$$D_x(\sec x) = \sec x \tan x \quad D_x(\csc x) = -\csc x \cot x$$
2.4 (continued)

Ex 1. Find $y'$ for the following functions.

(a) $y = \sin^2 x = (\sin x)(\sin x)$

(b) $y = \cot x$

(c) $y = \frac{\cos x + \sin x}{x^2 + 1}$

(d) $y = \sin^2 x + \cos^2 x$
Ex2 Find the tangent line to \( y = \cot x \) at \( x = \frac{\pi}{4} \).
2.5 The Chain Rule

The Chain Rule
\[ D_x(f(g(x))) = f'(g(x))g'(x) \]

or \[ D_x y = (D_x y)(D_x u) \]

Basically, we differentiate from the "outside-in."

This is very useful if we need to differentiate something like \( f(x) = 3(x^2 - 2x + 1)^8 \) and you really don't want to multiply it out 8 times.

Ex 1 If \( y = (3x^3 - 4x + 5)^{10} \), find \( y' \).

Ex 2 \[ y = \frac{4}{(2x^2 + 6x^2)^5} \] find \( y' \).
2.5 (continued)

Ex 3. Find \( f'(x) \).

(a) \( f(x) = \sin^3(x^2 - 4x) \)

(b) \( f(x) = \left( \frac{2x+1}{x-5} \right)^4 \)

(c) \( f(x) = \cos^2(\cos(\cos x)) \)

(d) \( f(x) = \sin^2(4x)(2x^5 + 3x^3 - 1)^3 \)
We can think of the chain rule as
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.
\]

Ex 4. Find \( \frac{dy}{dx} \).

(a) \( y = [(2x^2 + 3) \cos x]^4 \)

(b) \( y = (-3x + \frac{5}{x})^{-4} \)

(c)
### 2.6 Higher-Order Derivatives

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<td>n^th</td>
<td>f^(n)(x)</td>
<td>y^(n)</td>
<td>D_x^n y</td>
<td>(\frac{d^n y}{dx^n})</td>
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To get the second derivative, you need the first derivative first. It's a recursive process.

**Ex 1** Find \(f''(x)\) for \(f(x) = (3-5x)^5\)
2.6 (continued)

Ex 2  Find \( \frac{d^2 y}{dx^2} \) for \( y = x \sin \left( \frac{1}{x} \right) \).

Ex 3  What is \( D_x^5 (3x^4 - 2x^3 + x^2 - 7) \)?

Ex 4  Find a formula for \( D_x^n \left( \frac{1}{x} \right) \).
We know \( a(t) = v'(t) \), i.e., acceleration = derivative of velocity with respect to time.

\[ a(t) = v'(t) = s''(t) \]

**Ex 5 (No 24)** An object moves along a horizontal coordinate line according to \( s(t) = t^3 - 6t^2 \). \( s = \) directed distance from origin (in ft), \( t = \) time (in seconds).

(a) What are \( v(t) \) and \( a(t) \)?

(b) When is the object moving to the right?

(c) When is it moving to the left?
2.6 (continued)

(d) When is its acceleration negative?

(e) Draw a schematic diagram that shows the motion of the object.
Given \( 2y^3 - y^2 = x^3 + 5 \)

This is a function \( y \equiv f(x) \), i.e. \( y(x) \), given implicitly (because we cannot solve directly for \( y \) in terms of \( x \)). So, how do we differentiate it? We start w/ the given eqn. + differentiate both sides.

\[
\frac{d}{dx}(2y^3 - y^2) = \frac{d}{dx}(x^3 + 5)
\]

\[
(\Rightarrow) \quad \frac{d}{dx}(2y^3) - \frac{d}{dx}(y^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(5)
\]

\[
(\Rightarrow) \quad 6y^2 y' - 2y y' = 3x^2
\]

\[
(\Rightarrow) \quad y' (6y^2 - 2y) = 3x^2
\]

\[
(\Rightarrow) \quad y' = \frac{3x^2}{6y^2 - 2y}
\]

So, we now have the derivative \( y' = \frac{dy}{dx} \) of our function. Notice that it is given in terms of \( x \) and \( y \).
2.7 (continued)

Let's check to make sure implicit differentiation is reasonable.

If we have $x^2 + 2xy + 3y = 0$, we can differentiate it two ways.

1. **Implicit**
   
   $2x + 4xy + 2x^2y' + 3y + 3xy' = 0$
   
   $2x^2y' + 3xy' = -2x - 4xy - 3y$
   
   $y'(2x^2 + 3x) = -2x - 4xy - 3y$
   
   $y' = \frac{-2x - 4xy - 3y}{2x^2 + 3x}$

2. **Explicit**
   
   $x^2 + 2xy + 3y = 0$
   
   $(\Rightarrow) \quad y(2x^2 + 3x) = -x^2$
   
   $(\Rightarrow) \quad y = \frac{-x^2}{2x^2 + 3x}$
   
   $\Rightarrow y' = \frac{(2x^2 + 3x)(-2x) + x^2(4x + 3)}{(2x^2 + 3x)^2}$

Are they the same?
Ex 1 Find \( \frac{dy}{dx} \) for the following functions.

(a) \( x \sqrt{y+1} = xy + 1 \)

(b) \( 9x^2 + 4y^2 = 36 \)
Ex 2: Find the eqn of the tangent line at the indicated x-value.

(a) \( y + \cos(xy^2) + 3x^2 = 4 \) at \( x = 1 \)

(b) \( \sqrt{y} + xy^2 = 5 \) at \( x = 1 \)
Power Rule (revisited)

Let \( r \in \mathbb{Q} \). Then for \( x > 0 \):

\[
D_x (x^r) = rx^{r-1}.
\]

Why?

Let \( y = x^r = x^{p/q} \)

\[\Rightarrow y^{q} = x^{p}\]

\[
y^{q-1}y' = px^{p-1}
\]

\[
y' = \frac{p}{q} \left( \frac{x^{p-1}}{y^{q-1}} \right)
\]

\[
y' = \frac{p}{q} \left( \frac{x^{p-1}}{(x^{p/q})^{q-1}} \right)
\]

\[
y' = \frac{p}{q} \left( \frac{x^{p-1}}{x^{p-1/q}} \right)
\]

\[
y' = r \left( x^{p-1-p+q/q} \right)
\]

\[
y' = r \left( x^{q/q-1} \right)
\]

\[
y' = r x^{r-1} \]

**Ex 3**

Find \( y' \) if \( y = \sqrt[3]{x} - 2x^{3/2} \)
2.8. Related Rates

Remember that derivatives are rates. These problems are basically story problems that show the relevance of using derivatives. 

**EX 1** Assuming that a soap bubble retains its spherical shape as it expands, how fast is its radius increasing when its radius is 3 inches if air is blown into it at a rate of 3 cubic inches per second?

\[ V = \frac{4}{3} \pi r^3 \]

volume of a sphere

\[
\frac{dV}{dt} = 3 \text{ in}^3/\text{sec}
\]

We want to know \( \frac{dr}{dt} \) when \( r = 3 \text{ in} \).

And we're given that \( \frac{dV}{dt} = \frac{3 \text{ in}^3}{\text{sec}} \).

\[ \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \]

\[ \Rightarrow \frac{dr}{dt} = \frac{dV}{dt} \left( \frac{1}{4\pi r^2} \right) \]

\[ \Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2} \] and when \( r = 3 \), \( \frac{dr}{dt} = \frac{3}{4\pi (3^2)} = \frac{1}{12\pi} \)

\[ \Rightarrow \frac{dr}{dt} = \frac{1}{12\pi} \approx 0.0265 \text{ in/sec} \]
Ex 2. A child is flying a kite. If the kite is 90 ft above the child's hand level and the wind is blowing it on a horizontal course at 5 ft/sec, how fast is the child letting out the cord when 150 ft of cord is out? (Assume that the cord remains straight from hand to kite...not very realistic.)

We want to know \( \frac{dh}{dt} \) when \( h = 150 \) ft.

\[
\frac{dx}{dt} = 5 \text{ ft/sec}
\]
Ex3 A student is using a straw to drink from a conical paper cup, whose axis is vertical, at a rate of 3 cm³ per second. If the height of the cup is 10 cm, and the diameter of its opening is 6 cm, how fast is the level of the liquid falling when the depth of the liquid is 5 cm?
2.9 Differentials + Approximations

We've seen the notation \( dy \) and we've never separated the symbols. Now, we'll give meaning to \( dy + dx \).

We know \( \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0) \) which gives the derivative (slope) of the function \( f(x) \) at \( x = x_0 \).

If \( \Delta x \) is really small, then

\[
\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \approx f'(x_0)
\]

\[
\Rightarrow \quad f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \Delta x
\]

\( \Delta y \) actual change in \( y \) from \( x_0 \) to \( x_0 + \Delta x \)

\( dy \) an approximation of the change in \( y \).

**Defn: Differentials**

Let \( y = f(x) \) be a differentiable function of \( x \).

\( \Delta x \) is an arbitrary increment of \( x \).

\( dy \) is actual change in \( y \) as \( x \) goes from \( x \) to \( x + \Delta x \) i.e. \( dy = f(x + \Delta x) - f(x) \).

\( \Delta y \) is actual change in \( y \) from \( x_0 \) to \( x_0 + \Delta x \) i.e. \( \Delta y = f(x_0 + \Delta x) - f(x_0) \).

\( dy = f'(x) \Delta x \) (\( dy \) is called differential of \( y \))
Ex 1 Find dy.

(a) \( y = 4x^3 - 2x + 5 \)
\[ dy = (12x^2 - 2) \, dx \]

(b) \( y = 2\sqrt{x^4 + 6x} \)

(c) \( y = \cos(x^3 - 5x + 11) \)

(d) \( y = (x^{10} + \sqrt{\sin 2x})^2 \)
2.9 (continued)

Differentials can be used for approximations!

If \( f(x+\Delta x) - f(x) \approx f'(x) \Delta x \), then

\[ f(x+\Delta x) \approx f(x) + f'(x) \Delta x \]

**Ex 2** Find a good approximation for \( \sqrt{9.2} \) (w/o a calculator ^\( \dagger \)).

Let \( f(x) = \sqrt{x} \), when \( x = 9 \), \( f(x) = 3 \). Let \( \Delta x = 0.2 \).

\[ f(x+\Delta x) \approx f(x) + f'(x) \Delta x \] becomes

\[ f(9.2) \approx f(9) + f'(9)(0.2) \] and we know

\[ f'(x) = \frac{1}{2} x^{-1/2} \implies f'(9) = \frac{1}{2}(9^{-1/2}) = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6} \]

\[ f(9.2) \approx \sqrt{9} + \frac{1}{6}(0.2) = 3 + \frac{0.1}{3} = 3 + 0.0333 \]

\[ \approx 3.03 \] i.e. \( \sqrt{9.2} \approx 3.03 \)

**Ex 3** Use differentials to approximate the increase in the surface area of a soap bubble when its radius increases from 4 inches to 4.1 inches.

\[ A = 4\pi r^2 \]
Ex 4: The height of a cylinder is measured as 12 cm with a possible error of ±0.1 cm. Evaluate the volume of the cylinder with radius 4 cm and give an estimate for the possible error in this value.

\[ V = \pi r^2 h = \pi (4)^2 h \]

\[ \Rightarrow V = 16\pi h \]