2.1 Two Problems W/ One Theme

Archimedes - slope of a tangent line
Kepler/Galileo/Newton - instantaneous velocity

- \( Q = \) "movable" pt
- \( P = \) pt in question
- Secant line = line thru \( P + Q \).
- Tangent line = limiting position (if it exists) of secant line as \( Q \) moves thru \( P \) along the curve.

Slope of secant line
\[
m = \frac{f(c+h) - f(c)}{c+h - c} = \frac{f(c+h) - f(c)}{h}
\]

Slope of tangent line
\[
m = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}
\]
2.1 (continued)

Ex 1 Find eqn of tangent line to \( y = \frac{2}{x} \) at \( x = 1 \).

Ex 2 Find slope of \( y = -x^2 + 3x \) when \( x = -1, 2, 5 \).
If it takes me 6 hrs to drive 400 miles, then my avg. velocity is \( \frac{400}{6} \approx 67 \text{ mi/hr} \).

But surely I didn’t drive that speed the whole time.

\[ V_{\text{avg}} = ? \] for different time intervals

<table>
<thead>
<tr>
<th>( t_{\text{start}} )</th>
<th>( t_{\text{end}} )</th>
<th>( V_{\text{avg}} )</th>
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<tbody>
<tr>
<td>2</td>
<td>3</td>
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<td>2</td>
<td>2.5</td>
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\[ \Rightarrow \text{Velocity at time } t = 2 \text{ hrs } = \frac{d_{\text{end}} - d_{\text{start}}}{t_{\text{end}} - t_{\text{start}}} \]

\( \text{Math1210} \) (38)
2.1 (Continued)

- Geometrically finding slope of tangent line to a curve is exactly the same mathematical calculation as finding the instantaneous velocity for a moving object.

EX 3 (#14) An object travels along a line so that its position $s$ is given by $s(t) = t^2 + t$ (measured in meters, $t$ measured in seconds).

(a) What is its avg velocity on interval $2 \leq t \leq 3$?

(b) Avg velocity on $2 \leq t \leq 2.003$?

(c) Avg velocity on $2 \leq t \leq 2.004$?

(d) Instantaneous velocity at $t = 2$?

* "rate of change" means instantaneous rate of change
2.2 The Derivative

**Defn Derivative**

The derivative of $f$ is another function $f'$

$$\exists \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \forall x$$

provided the limit exists and is finite for some $x$-value.

If $f'(c)$ exists, we say $f(x)$ is differentiable at $x=c$.

**Ex 1** Find $f'(x)$ given $f(x) = 2\sqrt{x-1}, \quad x \geq 1$
2.2 (continued)

Another form of definition of derivative

\[ f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \]

Ex 2: Use above definition of \( f' \) to find \( h'(c) \)

if \( h(x) = \frac{3}{x-5} \).
Ex 3 Each of these is a derivative for some function. Can you find the function?

(a) \[ f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 - 2(x+h) - (3x^2 - 2x)}{h} \]

(b) \[ \lim_{x \to 3} \frac{\frac{4}{x} - \frac{4}{3}}{x - 3} \]

Ex 4 Let \( f(x) = |x| \)

Try to find \( f'(0) \):

\[
\begin{align*}
f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h \to 0} \frac{|0+h| - |0|}{h} \\
&= \lim_{h \to 0} \frac{|h|}{h} \\
&= \begin{cases} 
1 & \text{if } h \to 0^+ \\
-1 & \text{if } h \to 0^- 
\end{cases}
\]

\[ \Rightarrow f'(0) \text{ DNE} \]
2.2 (continued)

Visually, we can see a pt where the derivative (slope) DNE by looking for "corners" or vertical tangents, in the graph of the function.

What can we say about derivative of this function at \( x=3, 7 \) and \( 10 \)?

**Then**

Differentiability \( \Rightarrow \) Continuity

If \( f'(c) \) exists, then \( f \) is continuous at \( x=c \).

Also if \( f(x) \) is discontinuous \( x=c \), then \( f'(c) \) DNE.
2.3 Rules for Finding Derivatives

**Constant Function Rule**
If \( f(x) = k \), \( k \in \mathbb{R} \), \( f'(x) = 0 \) (or \( D_x(k) = 0 \)).

**Identity Rule**
If \( f(x) = x \), then \( f'(x) = 1 \) (or \( D_x(x) = 1 \)).

**Power Rule**
If \( f(x) = x^n \), \( n \in \mathbb{Z}^+ \), \( f'(x) = nx^{n-1} \) (or \( D_x(x^n) = nx^{n-1} \)).

**Constant Multiple Rule**
If \( k \in \mathbb{R} \), \( f'(x) \) exists, then \( D_x[kf(x)] = k(D_x(f(x))) \).

**Sum and Difference Rule**
If \( f' \) and \( g' \) exist, then
\[
D_x[f(x) \pm g(x)] = D_xf(x) \pm D_xg(x)
\]

**Example 1**
Find \( f'(x) \) if \( f(x) = 3x^3 - 4x^2 + x^5 + 2x^3 - x^2 + 4 \)
2.3 (continued)

**Product Rule**

If \( f \) and \( g \) are differentiable, then

\[
\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)]
\]

i.e. \((fg)' = f'g + g'f = g'f + f'g\)

**Ex 2** Find \( f''(x) \) for \( f(x) = (2x^3 - 4x + 1)(3x + 5) \)

Use product rule:

Multiply out and use power rule to check:
Quotient Rule

Let \( f \) and \( g \) be differentiable functions, \( g(x) \neq 0 \).

Then \[
D_x \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) D_x [f(x)] - f(x) D_x [g(x)]}{g^2(x)}
\]

E.g. \[
\left( \frac{f}{g} \right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}
\]

I usually remember this one a bit differently.

If \( f = \frac{\text{high}}{\text{low}} \), then \( f' = \frac{\text{low} \cdot d(\text{high}) - \text{high} \cdot d(\text{low})}{\text{low}^2} \)

"\text{low} d \text{ hi minus hi } d \text{ low over low squared}"

Ex 3 Find \( f'(x) \) if \( f(x) = \frac{2x^2 + 4x - 1}{3x - 2} \)
Ex 4 \[ y = \frac{-3}{x} + \frac{2}{x^4 - 7x} \] Find \( y' \).

\[ \Rightarrow \quad D_x(x^{-n}) = -n x^{-(n+1)} \]

because \[ f(x) = x^{-n} = \frac{1}{x^n} \] \( \Rightarrow \) \[ f'(x) = \frac{x^{n}(0) - 1(x^{n}n^{-1})}{x^{2n}} \]

\[ = \frac{-nx^{n-1}}{nx^{2n}} = -nx^{n-2n} \]

\[ = -nx^{n-1} \]

\[ = -nx \]

i.e. the Power Rule is true for \(-ve\) integers, too!
Ex 5  Find $f'(x)$ if $f(x) = \frac{5x-y}{3x^2+1}$

Ex 6  Find $y'$ if $y = 3x(x^3-2x+1)$

Ex 7  Find $g'(x)$ if $g(x) = \frac{-3}{x^5} + \frac{2}{x}$
0.7 Trigonometric Fns

Unit circle (r=1) centered at (0,0)

\[ \cos \theta = x \quad \sin \theta = y \]

-1 ≤ \sin \theta ≤ 1
-1 ≤ \cos \theta ≤ 1

\[ \sin \theta = \frac{O}{H} \]
\[ \cos \theta = \frac{A}{H} \]

From Pythagorean Theorem, we know

\[ A^2 + O^2 = H^2 \] (for above triangle)

(\Rightarrow) \quad \frac{A^2}{H^2} + \frac{O^2}{H^2} = 1 \quad \text{(divide both sides by } H^2)\]

(\Rightarrow) \quad \left( \frac{A}{H} \right)^2 + \left( \frac{O}{H} \right)^2 = 1

(\cos \theta)^2 + (\sin \theta)^2 = 1

or we write it this way

\[ \cos^2 \theta + \sin^2 \theta = 1 \]
Other Trig Fns

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} \]
\[ \cot \theta = \frac{\cos \theta}{\sin \theta} \]
\[ \sec \theta = \frac{1}{\cos \theta} \]
\[ \csc \theta = \frac{1}{\sin \theta} \]

If we look at \( \sin^2 \theta + \cos^2 \theta = 1 \), then

\[ \frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \quad (\Rightarrow) \]
\[ 1 + \cot^2 \theta = \csc^2 \theta \]

Also

\[ \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \quad (\Rightarrow) \]
\[ \tan^2 \theta + 1 = \sec^2 \theta \]

Other Trig Properties

1. \( \sin \theta = \sin (\theta + 2n\pi) \quad \forall \theta \in \mathbb{R} \)
2. \( \cos \theta = \cos (\theta + 2n\pi) \quad \forall \theta \in \mathbb{R} \)

\( \sin \theta \) is an odd fn, i.e. \( -\sin \theta = \sin (-\theta) \)

\( \cos \theta \) is an even fn, i.e. \( \cos \theta = \cos (-\theta) \)

(You can see this is true on the unit circle.)

3. \( \sin \left( \frac{\pi}{2} - \theta \right) = \cos \theta \quad + \)
4. \( \cos \left( \frac{\pi}{2} - \theta \right) = \sin \theta \)
Graphs of $y = \sin x$ and $y = \cos x$

**Amplitude**: half the distance between lowest to highest height of graph

**Period**: the smallest $p \geq f(x+p)=f(x)$ for some $f(x)$.

For $\sin x$ and $\cos x$, the period is $2\pi$, i.e. every $2\pi$ interval (or $x$-axis), the curve repeats itself.
\[ f(x) = a \sin(b(x+c)) + d \]

Likewise for
\[ g(x) = a \cos(b(x+c)) + d \]

Amplitude

Vertical shift \( d \) units

Period of curve is \( \frac{2\pi}{b} \)

Horizontal shift \( c \) units

\[ 180^\circ = \pi \text{ (radians)} \]

(\# Note Trig properties in nice box on pg 47!)

Ex1 (a) Convert \( \frac{-\pi}{3} \) to degrees.

(b) Convert \( \frac{3\pi}{18} \) to degrees.

(c) Convert \(-120^\circ\) to radians.

(d) Convert \(600^\circ\) to radians.
0.7 (continued)

Ex 2
Evaluate (by calculator).

(a) \( \tan \left( \frac{\pi}{6} \right) \)

(b) \( \sec \left( \frac{\pi}{3} \right) \)

(c) \( \csc \left( \frac{\pi}{4} \right) \)

(d) \( \cos \left( \frac{\pi}{3} \right) \)

(e) \( \cot \left( \frac{5\pi}{6} \right) \)

(f) \( \sin \left( \frac{5\pi}{4} \right) \)
Ex 3. For the following functions, list the amplitude, period, horizontal shift, and then graph:

(a) \( y = 3 \cos(x + \frac{\pi}{2}) - 1 \)

(b) \( y = 2 \sin(x + \frac{\pi}{4}) \)

(c) \( y = \frac{1}{2} \cos(2(x + \frac{\pi}{2})) + 3 \)
Ex 4. Verify the identity

\[ \cos(3x) = 4 \cos^3 x - 3 \cos x \]
\[ \cos^2 \left( \frac{\pi}{12} \right) = \cos^2 \left( \frac{\pi}{6} \right) \]

We know \( \cos \left( \frac{x}{2} \right) = \pm \sqrt{\frac{1 + \cos x}{2}} \)

\[ \Rightarrow \cos^2 \left( \frac{x}{2} \right) = \frac{1 + \cos x}{2} \]
1.4. Limits Involving Trig Funcs

**Theorem**
If $c \in \mathbb{R}$ is in the function's domain,

\[
\begin{align*}
\lim_{x \to c} \sin x &= \sin c \\
\lim_{x \to c} \csc x &= \csc c \\
\lim_{x \to c} \cos x &= \cos c \\
\lim_{x \to c} \sec x &= \sec c \\
\lim_{x \to c} \tan x &= \tan c \\
\lim_{x \to c} \cot x &= \cot c
\end{align*}
\]

i.e. Basically, we can still just plug in $x = c$, if it works.

<table>
<thead>
<tr>
<th>Special Trig limits</th>
<th>Theorem</th>
</tr>
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<tbody>
<tr>
<td>1. $\lim_{t \to 0} \frac{\sin t}{t} = 1$</td>
<td>2. $\lim_{t \to 0} \frac{1 - \cos t}{t} = 0$</td>
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</tbody>
</table>

**Proof (of 2)**

\[
\lim_{t \to 0} \frac{1 - \cos t}{t} = \lim_{t \to 0} \frac{(1 - \cos t)(1 + \cos t)}{t(1 + \cos t)}
\]

\[
= \lim_{t \to 0} \frac{1 - \cos^2 t}{t(1 + \cos t)} = \lim_{t \to 0} \frac{\sin^2 t}{t(1 + \cos t)}
\]

\[
= \left[ \lim_{t \to 0} \frac{\sin t}{t} \right] \cdot \left[ \lim_{t \to 0} \frac{\sin t}{1 + \cos t} \right] = 1 \cdot 0 = 0
\]
1.4 (Continued)

Ex 1 \( \lim_{{x \to 0}} \frac{3x + \tan x}{\sin x} \)

Ex 2 \( \lim_{{t \to 0}} \frac{\sin^2 3t}{2t} \)
Ex 3 \[ \lim_\theta \to 0 \frac{\tan(5\theta)}{\sin(2\theta)} \]

(Hint: \( \sin 5\theta = \sin(3\theta + 2\theta) = \sin 3\theta \cos 2\theta + \cos 3\theta \sin 2\theta \)
and \( \sin 3\theta = \sin(\theta + 2\theta) = \sin \theta \cos 2\theta + \cos \theta \sin 2\theta \))

\[ \lim_\theta \to 0 \frac{\tan 5\theta}{\sin 2\theta} = \lim_\theta \to 0 \frac{\sin 5\theta}{\sin 2\theta} \left( \frac{1}{\cos 5\theta} \right) \]

=