Key Definitions: Sections 2.1-2.4

- The identity matrix $I_n$ is

- A diagonal matrix is

- A zero matrix is

- The transpose of a matrix $A$ is

- An elementary matrix is

- A partitioned or block matrix $A$ is
Major Theorems: Sections 2.1-2.4

Section 2.1

<table>
<thead>
<tr>
<th>Theorem 1 Properties of Matrix Addition and Scalar Multiplication Let $A$, $B$, and $C$ be matrices of the same size, $m \times n$, and let $r$ and $s$ be scalars.</th>
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</thead>
<tbody>
<tr>
<td>(a) $A + B =$</td>
</tr>
<tr>
<td>(b) $(A + B) + C =$</td>
</tr>
<tr>
<td>(c) $A + 0 =$</td>
</tr>
<tr>
<td>(d) $r(A + B) =$</td>
</tr>
<tr>
<td>(e) $(r + s)A =$</td>
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<tr>
<td>(f) $r(sA) =$</td>
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<tr>
<th>Theorem 2 Properties of Matrix Multiplication Let $A, B,$ and $C$ be matrices and $r$ be a scalar such that the sums and products below are defined. Then,</th>
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<tbody>
<tr>
<td>(a) $A(BC) =$</td>
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<tr>
<td>(b) $A(B + C) =$</td>
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<tr>
<td>(c) $(B + C)A =$</td>
</tr>
<tr>
<td>(d) $r(AB) =$</td>
</tr>
<tr>
<td>(e) $I_mA =$</td>
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<tr>
<th>Theorem 3 Transpose Properties Let $A$ and $B$ be matrices whose sizes are appropriate for the following sums and products.</th>
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<tr>
<td>(a) $(A^T)^T =$</td>
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<tr>
<td>(b) $(A + B)^T =$</td>
</tr>
<tr>
<td>(c) For any scalar $r$, $(rA)^T =$</td>
</tr>
<tr>
<td>(d) $(AB)^T =$</td>
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</table>

Note: The transpose of a product of matrices equals the product of their transposes in reverse order.
Theorem 4 2 × 2 Inverses Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). If \( ad - bc \neq 0 \), then \( A \) is invertible, and

\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

If \( ad - bc = 0 \), then \( A \) is not invertible (is singular).

The quantity \( ad - bc \) is called the \textbf{determinant} of \( A \).

Thus, a 2 × 2 matrix is invertible if and only if \( \det A \neq 0 \).

Theorem 5 Matrix Equation Solutions and Inverses If \( A \) is an invertible \( n \times n \) matrix, then for each \( b \in \mathbb{R}^n \), the equation \( Ax = b \) has the unique solution, \( x = A^{-1}b \).

Theorem 6 Properties of Inverses

(a) If \( A \) is an invertible matrix, then \( A^{-1} \) is invertible and

\[
(A^{-1})^{-1} = \quad \quad
\]

(b) If \( A \) and \( B \) are \( n \times n \) invertible matrices, then \( AB \) is also invertible. The inverse of \( AB \) is the product of the inverses of \( A \) and \( B \) in reverse order. That is,

\[
(AB)^{-1} = \quad \quad
\]

(c) If \( A \) is an invertible matrix, then \( A^T \) is also invertible. The inverse of \( A^T \) is the transpose of \( A^{-1} \). That is,

\[
(A^T)^{-1} = \quad \quad
\]

Theorem 7 An \( n \times n \) matrix \( A \) is invertible if and only if \( A \) is row equivalent to \( I_n \), and in this case, any sequence of elementary row operations that reduces \( A \) to \( I_n \) also transforms \( I_n \) to \( A^{-1} \).
Theorem 8 The Invertible Matrix Theorem Let $A$ be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given $A$, the statements are either all true or all false.

(a) $A$ is an invertible matrix.
(b) $A$ is row equivalent to the $n \times n$ matrix.
(c) $A$ has _______ pivots.
(d) The equation $Ax = 0$ has only the _______ solution.
(e) The columns of $A$ form a linearly _______ set.
(f) The linear transformation $x \mapsto Ax$ is _______.
(g) The equation $Ax = b$ has _______ solution for each $b$ in $\mathbb{R}^n$.
(h) The columns of $A$ _______ $\mathbb{R}^n$.
(i) The linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ _______ $\mathbb{R}^n$.
(j) There is an $n \times n$ matrix $C$ such that $CA = _______.$
(k) There is an $n \times n$ matrix $D$ such that $AD = _______.$
(l) $A^T$ is an _______ matrix.

Theorem 9 Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let $A$ be the standard matrix for $T$. Then, $T$ is invertible if and only if $A$ is an invertible matrix, and the linear transformation $S$ given by $S(x) = A^{-1}x$ is the unique function such that

\begin{align*}
    A^{-1}(Ax) &= S(T(x)) = x \text{ for all } x \in \mathbb{R}^n \\
    A(A^{-1}x) &= T(S(x)) = x \text{ for all } x \in \mathbb{R}^n
\end{align*}

where $S = T^{-1}$ is the inverse of $T$. 
Supplemental Practice Problems:

1. Compute the inverse of \( A = \begin{bmatrix} 1 & 0 & 5 \\ 4 & 2 & 20 \\ 0 & 4 & -5 \end{bmatrix} \) using the inverse algorithm, \([A \ I] \sim \begin{bmatrix} I & A^{-1} \end{bmatrix}\).

2. Find the inverse of the following matrices:
   
   (a) \( \begin{bmatrix} -1 & 2 & 2 \\ 2 & -4 & -3 \\ -1 & 1 & 4 \end{bmatrix} \)
   
   (b) \( \begin{bmatrix} 5 & 5 & 2 \\ 4 & 5 & 2 \\ -2 & 1 & 0 \end{bmatrix} \)

3. Consider the matrix \( A = \begin{bmatrix} 2 & 3 & 1 \\ -2 & 5 & 2 \\ 0 & 3 & 1 \end{bmatrix} \)
   
   (a) Show that \( A^{-1} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ -1 & -1 & 3 \\ 3 & 3 & -8 \end{bmatrix} \).
   
   (b) Using matrix \( A \) or \( A^{-1} \), determine the number of pivots of \( A \) and whether the columns of \( A \) are linearly independent or dependent.

4. Suppose that an \( n \times n \) matrix has a column which is a multiple of another column. Either give an example of an invertible matrix of this type or explain why such a matrix is not invertible.

5. Consider the following matrices
   
   \( A = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 2 & 1 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & -2 \\ 2 & 5 \\ 3 & 3 \end{bmatrix} \), \( C = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \).

   Calculate (if possible) each of the following matrix products:
   
   (a) \( AB = \begin{bmatrix} 12 & 3 \\ 4 & 19 \end{bmatrix} \)
   
   (b) \( BA = \begin{bmatrix} 8 & -4 & 2 \\ -11 & 8 & 13 \\ -3 & 3 & 15 \end{bmatrix} \)
   
   (c) \( AC \) DNE
   
   (d) \( CA = \begin{bmatrix} 19 & -12 & 3 \\ -11 & 7 & -1 \end{bmatrix} \)

   # cols of \( A \) \# rows of \( C \)

   \# cols of \( A \) \# rows of \( C \)
6. Let \( R \) be the rectangle with vertices \((-2, -1), (-2, 2), (3, 2), (3, -1)\). Consider the linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by \( T(x_1, x_2) = (4x_1 - 3x_2, -x_1 + x_2) \).

(a) Find the standard matrix \( A \) of the linear transformation \( T \), and sketch the image of the rectangle \( R \) under \( T \).

(b) Find the standard matrix \( A^{-1} \) corresponding to the inverse of \( T \) and sketch the image of the rectangle \( R \) under \( T^{-1} \).

7. Let \( R \) be the rectangle with vertices \((-2, -1), (-2, 2), (3, 2), (3, -1)\). Consider the linear transformation \( S : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) which maps the unit square to the parallelogram pictured below.

(a) Find the standard matrix \( B \) associated to \( S \) and sketch the image of the rectangle \( R \) under \( S \).

(b) Find the matrix \( B^{-1} \) associated to the inverse of \( S \) and sketch the image of the rectangle \( R \) under \( S^{-1} \).

8. Consider the matrices \( A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & b \\ c & -1 \end{bmatrix} \) where \( b \) and \( c \) are unknowns. Find values of \( b \) and \( c \) such that \( AB = BA \).

9. Given \( \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X & Y \\ Z & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \). Find formulas for \( X \), \( Y \) and \( Z \) in terms of \( A \), \( B \) and \( C \).

Justify your calculations. That is, in some cases, you may need to make assumptions about the size of a matrix in order to produce a formula.

10. Let \( A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \) where \( B \) and \( C \) are square blocks. Show that \( A \) is invertible if and only if both \( B \) and \( C \) are invertible.

\[
(\Rightarrow) \text{ Assume } A^{-1} \text{ exists. Let } A^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \text{ and we know } AA^{-1} = I.

\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
\]
$1. \ BE = I \quad 2. \ CH = I \quad 3. \ BF = 0 \ \Rightarrow \ EF = 0$

$\Rightarrow \ E = B^{-1} \quad \Rightarrow \ H = C^{-1} \quad 4. \ CG = 0 \quad G = 0$

$(\Leftarrow)$ Assume $B^{-1}, C^{-1}$ exists. Hence $A^{-1}$ exists. Let $D = \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}$.

$AD = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$

$\Rightarrow D = A^{-1}$

**Problem 6**

$(-3, -1) \quad (-2, 2) \quad (3, 2) \quad (3, -1)$

$T(x_1, x_2) = (4x_1 - 3x_2, -x_1 + x_2)$

(a) $T(-2, -1) = (4(-2) - 3(-1), 2 + 1)$

$= (-5, 1)$

$T(-2, 2) = (-8 - 6, 2 + 2) = (-14, 4)$

$T(3, 2) = (12 - 6, -3 + 2) = (6, -1)$

$T(3, -1) = (12 + 3, -3 - 1) = (15, -4)$

$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix} = x_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
(c) \[ A = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{4-3} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \]

\[ A^{-1} \begin{bmatrix} 15 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 15 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \]

\[ A^{-1} \begin{bmatrix} -6 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -6 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \] etc. we'll undo

T by multiplying all the pts in the transformed quadrilateral by \( A^{-1} \), and get back to original pts.

#8) \[ \begin{bmatrix} 3+5c & 3b-5 \\ 1+2c & b-2 \end{bmatrix} = \begin{bmatrix} 3+10 & 5+2b \\ 3c-1 & 5c-2 \end{bmatrix} \]

1. \[ 3+5c = 3+10 \]
   \[ 5c = 10 \]
   \[ c = 2 \]

2. \[ 3b-5 = 5+2b \]
   \[ b = 10 \]

3. \[ 1+2c = 3c-1 \]
   \[ 2c = -2 \]
   \[ c = -1 \]

4. \[ b-2 = 5c-2 \]
   \[ b = 5c \]
   \[ b = 5 \]
   \[ b = 10 \]
   \[ 10-2 = 5(2)-2 \] check

\[ B = \begin{bmatrix} 1 & 10 \\ 2 & -1 \end{bmatrix} \]
\[
\begin{bmatrix}
A & B \\
0 & I
\end{bmatrix}
\begin{bmatrix}
x & y \\
0 & I
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\]

\[
\begin{bmatrix}
Ax & Ay & A^2+B \\
0 & 0 & I
\end{bmatrix} =
\begin{bmatrix}
I & 0 & 0 \\
0 & 0 & I
\end{bmatrix}
\]

\[
\Rightarrow Ax = I
\]

\[
\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}
\]

\[
A^2 = -B
\]

\[
A^2 + B = 0
\]

\[
A^2 = -B
\]

\[
\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}
\]

(Since \( A^2 = 0 \))

and \( A^{-1} \) exists

we had to assume \( A \) is square and \( A^{-1} \) exists.