# Divisibility by 9

Claim: If sum of digits of a # is divisible by 9, then so is the original #.

Pf: let $x$ = any whole number with digits $d_n d_{n-1} \ldots d_2 d_1 d_0$, where $d_i \in \{0, 1, \ldots, 9\}$ for all $i$.

For example, if $x = 4067$, then $d_3 = 4$, $d_2 = 0$, $d_1 = 6$, $d_0 = 7$.

Then $x$ can be written in expanded form as

$$x = d_n(10^n) + d_{n-1}(10^{n-1}) + d_{n-2}(10^{n-2}) + \ldots + d_2(10^2) + d_1(10) + d_0$$

(For our example, this would mean

$$x = 4067 = 4(10^3) + 0(10^2) + 6(10^1) + 7.)$$

If the sum of the digits adds to a multiple of 9, then we can write that mathematically as

$$d_0 + d_1 + d_2 + \ldots + d_{n-1} + d_n = 9m$$ for some whole number $m$.

$$\Rightarrow d_0 = 9m - d_1 - d_2 - \ldots - d_{n-1} - d_n$$

Now plug that in for $d_0$ in expanded form of $x$.

we get

$$x = 10^n d_n + 10^{n-1} d_{n-1} + \ldots + 10^2 d_2 + 10 d_1 + 9m - d_1 - d_2 - \ldots - d_n$$

$$\Leftrightarrow x = d_n(10^n - 1) + d_{n-1}(10^{n-1} - 1) + \ldots + d_2(10^2 - 1) + d_1(10 - 1) + 9m$$

But notice that

$$10-1=9$$
$$10^2-1=99$$
$$10^3-1=999$$
$$10^4-1=9999$$
$$\vdots$$
$$10^n-1=\underbrace{99\cdots9}_{n \text{ digits of } 9}$$

And all of these numbers are divisible by 9.

$$\Rightarrow x = 9\left[\underbrace{(11\cdots1)}_{n \text{ digits}} d_n + \underbrace{(11\cdots1)}_{n-1 \text{ digits}} d_{n-1} + \cdots + (11)d_2 + 1d_1 + m\right]$$

i.e. x is also divisible by 9.