5.3 Future Value of Annuities

Vocab annuity = financial plan characterized by regular payments

ordinary annuity vs. annuity due
payments made at end payments made at beginning of each
of each equal payment equal payment interval interval

Ex.1 Suppose you invest \$1000 at the end of each year for 5 years in an account that pays 10% interest compounded annually. What is the value after 5 yrs?

end of yr 1: \( P = 1000 \) compouds for \( t = 4 \) yrs
\[ S_1 = 1000 (1.1)^4 \]
end of yr 2:
end of yr 3:
end of yr 4:
end of yr 5:
5.3 (cont)

Generally, thus, for an ordinary annuity, the future value is

\[ S = \frac{R(1-(1+r_C)^{-n})}{r_C} \]

(sum of geometric sequence)

\[ S = R \left(1-(1+r_C)^{-n}\right) \]

Future value of ordinary annuity

\[ r_C = \frac{r}{n} \]

\[ R = \text{amt. deposited each period} \]

\[ n = nt \]

Ex 2 (Twins story)

Twin 1: At the end of college, she invests $2000 at the end of each year for 8 years in an account that earns 10% compounded annually. After 8 years, she contributes nothing, but it continues to earn 10% compounded annually for 36 more years. How much does she have then?
5.3 (cont)

Twice: At the end of college, he invests nothing for 8 years. Then he puts $2,000 into an account at the end of each year for 36 years, earning 10\% interest compounded annually. How much does he have then?

Ex 3: How much should be invested quarterly (at end of each qtr) at 12\% interest compounded quarterly to pay off a debt of $30,000 in 10 yrs?

\[
R = S\left(\frac{r_c}{(1+r_c)^N-1}\right)
\]

Sinking Fund

The payment that needs to be invested every period to pay off debt of $S$ at the end.
Ex 4. Find the future value of an account if $100 deposited at the beginning of each month for 5 yrs into an account that pays 8% compounded monthly.

\[
S = R \left[ \frac{(1+r_c)^{n+1}-1}{r_c} \right] - R
\]

\[
= R \left[ \frac{(1+r_c)^n - 1 - r_c}{r_c} \right]
\]

\[
= R \left[ \frac{(1+r_c)^n}{r_c} - (1+r_c) \right]
\]

\[
= R(1+r_c) \left[ \frac{(1+r_c)^n-1}{r_c} \right]
\]
5.4 Present Value of Annuity

PV of an annuity =) we calculate this when we leave a lump sum of $ in an account and make regular withdrawals (like what happens after a person retires)

Ordinary annuity vs. annuity due
- withdrawals occur at end of each period
- withdrawals occur at beginning of each period

Ex! You want to withdraw $1000 at the end of each year from an account that earns 10% interest compounded annually for 4 years. How much needs to be in the account from the start?

Compounding:
\[ S = P \left(1 + \frac{r}{n}\right)^{nt} \]
\[ r_e = \frac{r}{n}, \quad N = nt \]
\[ = S = P \left(1 + r_e\right)^N \]
\[ = P = S\left(1 + r_e\right)^{-N} \]

After 1st year: we need \( P_1 = 1000 \left(1 + 0.1\right)^{-1} = 1000 \left(1.1\right)^{-1} \)
After 2nd year: \( P_2 = 1000 \left(1.1\right)^{-2} \)
After 3rd year:
After 4th year:
\[ P_1 + P_2 + P_3 + P_4 = \]
(Sum of geometric sequence again!)
5.4 (cont)

<table>
<thead>
<tr>
<th>Present Value (PV) of ordinary annuity</th>
<th>PV of annuity due</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S = \frac{R(1 - (1+r)^{-n})}{r} )</td>
<td>( S_{due} = \frac{R(1+r)(1-(1+r)^{-n})}{r} )</td>
</tr>
</tbody>
</table>

**Ex 2** Find PV of annuity that pays $4000 at the end of each month from an account that earns 8% interest compounded monthly for 25 years.

**Ex 3** An inheritance of $500,000 will provide how much at the end of each year for 20 years, if money is worth 7.2% compounded annually?
Ex. 4 A lottery prize worth $1,800,000 is awarded in payments of $19,000 at the beginning of each month for 15 years. Suppose money is worth 6.67% monthly. What is the real value of the prize?

Deferred Annuity = when 1st payment is deferred until a later date at which pt regular payments are made.

\[ P = P_V \text{ of deferred annuity} \]
\[ m = \text{# periods of deferment} \]
\[ n = \text{# of regular withdrawals} \]
\[ R = \text{payment each withdrawal period} \]

\[ P = \frac{R(1-(1+r)^{-n})}{r(1+r)^m} \]
Ex5 Carol received a trust fund inheritance of $10,000 on her 30th birthday. She plans to use it to supplement her income with 20 quarterly payments beginning on her 60th birthday. If money is worth 8.19% compounded quarterly, how much will each payment be?
5.5 Loans and Amortization

_amortization = "installment loan"; a loan is
repaid by making all payments equal.

Bank is basically investing a lump sum of
$ and getting a periodic return which is exactly
like PV of ordinary annuity.

\[ R = S \left( \frac{r}{1-(1+r)^{-n}} \right) \]  

**Amortization Formula**

\[ S = \text{loan amount} \quad R = \text{payment amount} \]

Ex1 When you graduate college, you buy a new
car and can afford a monthly payment of $250.
If you get a special rate of 3.6% interest,
compounded monthly, for 6 years, how much
can you afford to borrow?
Ex. 2 Angie buys a house for $200,000. She puts $15,000 down and she gets a loan for the rest at 5.49% interest compounded monthly for 20 years. What will her payments be?

Amortization Schedule

A loan of $10,000 w/ interest rate of 10% could be repaid in 5 equal annual payments.

\[
R = \frac{10000 \left( \frac{0.1}{1 - (1.1)^{-5}} \right)}{5} = \$2637.97
\]

<table>
<thead>
<tr>
<th>Payment</th>
<th>Interest</th>
<th>Principal</th>
<th>Unpaid Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1(1000) = 1000</td>
<td>1637.97</td>
<td>8362.03</td>
</tr>
<tr>
<td>2</td>
<td>0.1(8362.03) = 836.20</td>
<td>1801.77</td>
<td>6560.24</td>
</tr>
<tr>
<td>3</td>
<td>0.1(6560.24) = 656.03</td>
<td>1981.94</td>
<td>4578.32</td>
</tr>
<tr>
<td>4</td>
<td>0.1(4578.32) = 457.83</td>
<td>2180.14</td>
<td>2398.18</td>
</tr>
<tr>
<td>5</td>
<td>0.1(2398.18) = 239.80</td>
<td>2398.18</td>
<td>0</td>
</tr>
</tbody>
</table>
Ex3 A company that buys a piece of equipment by borrowing $250,000 for 10 years at 6.7%, compounded monthly, has monthly payments of $2,775.51.

(a) Find the unpaid balance (loan payoff) after 1 year?

(b) During that first year, how much interest does the company pay?