9.1 Sequence and Series

Definitions/Vocabulary

Sequence ⇒ an ordered list of #s that follow a pattern; notation $\{a_n\}$

recursive formula ⇒ depends on previous term(s)

iterative formula ⇒ independent of previous terms; explicit.

term of sequence ⇒ a_i , $i = 1, 2, 3, \ldots, n$

finite sequence ⇒ a sequence that ends at some n.

infinite sequence ⇒ a sequence that never ends

Factorial ⇒ $n!$ (read "n factorial") is defined

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1)(n)$; and $0! = 1$

$n \in \mathbb{W}$.

Ex 1 $1, 1, 2, 3, 5, \ldots$ is Fibonacci sequence

(a) Is it infinite or finite?

(b) what is $a_1 = ?$ $a_2 = ?$ $a_6 = ?$

(c) write out next 4 terms of sequence.

(d) write formula (recursive) for a_n.

9.1 (cont)

Ex 2 (a) Write first five terms of sequence given by $a_n = (-1)^n \left(\frac{n}{n+1}\right)$. Is this formula recursive or iterative?

(b) Write first five terms of $a_n = \frac{1}{3} a_{n-1}$. Is formula recursive or iterative?

Ex 3 Write iterative & recursive formulas for a_n.

(a) $1, 2, \frac{2^2}{2}, \frac{2^3}{6}, \frac{2^4}{24}, \frac{2^5}{120}, \ldots$

(b) $2, -4, 6, -8, 10, \ldots$

Ex 4 Write the first 5 terms of the sequence. Use that pattern to write an iterative formula for a_n.

$$a_1 = 14 \quad , \quad a_{k+1} = (-2)a_k$$

Ex 5 Simplify.

(a) $\dfrac{26!}{23!}$

(b) $\dfrac{(3n+2)!}{(3n)!}$

9.1 (cont)

Series Defn

For sequence $\{a_n\}$ (i.e. $a_1, a_2, a_3, \ldots$), the n^{th} partial sum is $a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$.

The sum of all ∞ terms of sequence is an infinite series

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{i=1}^{\infty} a_i$$

Properties of Finite Sum operator (Σ) (it's a linear operator)

① $\sum_{i=1}^{n} (c a_i) = c \sum_{i=1}^{n} a_i$

② $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Ex 6 Find the sum $\sum_{k=1}^{4} (-1)^k (2k)$.

9.1 (cont)

Ex 7 Use sigma notation to write the sum.

$$\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} + \frac{120}{32} + \frac{720}{64}.$$

Ex 8 Find the sum of the infinite series

$$\sum_{i=1}^{\infty} 9\left(\frac{1}{10}\right)^{i}.$$

9.2 Arithmetic Sequences and Partial Sums

Defn An <u>arithmetic sequence</u> is a sequence whose successive terms have same difference. (i.e. we keep adding same # over + over again) (d = common difference)

$$a_n = a_{n-1} + d$$ (recursive formula)

$a_1 = a_1$

$a_2 = a_1 + d$

$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$

$a_4 =$

$a_5 =$

$\vdots$

$\boxed{a_n = \qquad\qquad}$ iterative formula for arithmetic sequence

<u>Sum of an arithmetic sequence</u> (finite)

Gauss' example

$\Rightarrow$ guess for formula $S_n =$

9.2 (cont)

$$\boxed{S_n = \frac{n}{2}(a_1 + a_n)}$$ sum of arithmetic sequence w/ n terms

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

Rearrange terms (since addition is commutative and associative).

$$a_1 + a_n = a_1 + a_1 + (n-1)d = 2a_1 + (n-1)d$$
$$a_2 + a_{n-1} = (a_1 + d) + (a_1 + (n-2)d) = 2a_1 + (n-1)d$$
$$a_3 + a_{n-2} = (a_1 + 2d) + (a_1 + (n-3)d) = 2a_1 + (n-1)d$$
$$\vdots$$

we have $\left(\frac{n}{2}\right)$ groups of this repeated sum.

$$\Rightarrow S_n = \left(\frac{n}{2}\right)(2a_1 + (n-1)d) = \left(\frac{n}{2}\right)(a_1 + a_n)$$

Ex 1 Are these sequences arithmetic? If so, find d.

(a) 5.3, 5.7, 6.1, 6.5, 6.9, ...

(b) $\ln 2, \ln 4, \ln 6, \ln 8, \ldots$

(c) 800, 400, 200, 100, 50, ...

Ex 2 Find a formula (iterative) for a_n, such that $\{a_n\}$ is arithmetic, and $a_1 = 0$, $d = -2/3$. Write the first five terms.

Ex 3 Find an iterative formula for this arithmetic sequence. $a_1 = 72$, $a_{k+1} = a_k - 6$.

Ex 4 If $a_1 = 3$, $a_2 = 13$, and $\{a_n\}$ is arithmetic, then $a_{50} = ?$

9.2 (cont)

Ex 5 Find the n^{th} partial sum of the arithmetic sequence.

(a) $2, 8, 14, 20, \ldots$ $n = 25$

(b) $a_1 = 15$, $a_{100} = 307$, $n = 100$

Ex 6 Find $\sum_{n=51}^{100} 7n$

9.3 Geometric sequences and Series

Defn A geometric sequence is a sequence whose successive terms have the same quotient, (i.e. we keep multiplying by same # repeatedly)

$a_n = a_{n-1} r$ (r = common ratio)

recursive formula

$a_1 = a_1$

$a_2 = a_1 r$

$a_3 = a_2 r = (a_1 r) r = a_1 r^2$

$a_4 =$

$a_5 =$

$\vdots$

$\boxed{a_n = \qquad\qquad\qquad}$

iterative formula for geometric sequence

Sum of a finite n geometric sequence

$$S_n = \sum_{i=1}^{n} a r^{i-1} = \sum_{i=0}^{n} a r^{i} = \frac{a(1-r^n)}{1-r}$$

Let $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

$\Leftrightarrow S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$

$\Rightarrow r S_n = a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + a_1 r^n$

$\Rightarrow S_n - r S_n = (a_1 + \cancel{a_1 r} + \cancel{a_1 r^2} + \cdots + \cancel{a_1 r^{n-2}} + \cancel{a_1 r^{n-1}})$
$\qquad - (\cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{a_1 r^3} + \cdots + \cancel{a_1 r^{n-1}} + a_1 r^n)$

$\Leftrightarrow S_n - r S_n = a_1 - a_1 r^n$

$S_n(1-r) = a_1(1-r^n) \Leftrightarrow \boxed{S_n = \frac{a_1(1-r^n)}{1-r}}$

n starts at 1

9.3 (cont)

Ex 1 Are these sequences geometric? If so, find r.

(a) $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \frac{4}{11}, \ldots$

(b) $9, -6, 4, -8/3, \ldots$

Ex 2 Write the first 5 terms of the geometric sequence w/ $a_1 = 6$, $r = -1/4$. Then, write iterative formula for a_n.

9.3 (cont)

Ex 3 Find an iterative formula for the geometric sequence $a_1 = 5, \quad a_{k+1} = 2a_k$.

Ex 4 If $a_2 = 3$, $a_5 = \frac{3}{64}$, find the 1st and 7th term of the geometric sequence.

Ex 5 Find the (finite) sum of the geometric sequence.

(a) $\sum_{n=0}^{40} 5\left(\frac{3}{5}\right)^n$

(b) $\sum_{n=1}^{100} 15\left(\frac{2}{3}\right)^{i-1}$

9.3 (cont)

Ex 6 Use summation notation to write the sum. $15 - 3 + \frac{3}{5} - \cdots - \frac{3}{625}$.

Infinite sum of Geometric sequence

We know $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = \frac{a_1(1-r^n)}{1-r}$.

If $|r| < 1$, then $r^n \to 0$.

$\Rightarrow \quad S = \sum_{i=1}^{\infty} a_1 r^{i-1} = \frac{a_1(1-0)}{1-r} = \frac{a_1}{1-r}$

$$\boxed{S = \sum_{i=1}^{\infty} a r^{i-1} = \sum_{i=0}^{\infty} a r^{i} = \frac{a}{1-r}}$$ if $|r| < 1$

Ex 7 Find the sum.

(a) $\sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^n$

9.3 (cont)

Ex 7 (cont)

(b) $\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$

(b) $1.3\overline{8}$ (Hint: $1.3\overline{8} = 1.3 + (0.08 + 0.008 + 0.0008 + \dots)$)