Math 1010 — Fall 2005 — Departmental Final Exam — Answers

-1- **(Fractions.)** Simplify:

$$\frac{\frac{3}{5} - \frac{2}{3}}{\frac{1}{3} + \frac{3}{5}} = \frac{\frac{9-10}{15}}{\frac{5+9}{15}} = \frac{-1}{15} \div \frac{14}{15} = \frac{-1}{15} \times \frac{15}{14} = -\frac{1}{14}.$$

-2- **(A Linear Equation.)** Solve the equation

$$4x - 3 = 10 - 2(x - 1)$$

Proceeding as usual we obtain

$$
\begin{array}{rcll}
4x - 3 & = & 10 - 2(x-1) & \text{distribute} \\
4x - 3 & = & 10 - 2x + 2 & \text{collect like terms} \\
4x - 3 & = & 12 - 2x & +2x + 3 \\
6x & = & 15 & \div 6 \\
x & = & \frac{15}{6} & \text{the solution} \\
 & = & \frac{5}{2} & \text{simplified}
\end{array}
\tag{1}
$$

Of course we check by substituting $x = \frac{5}{2}$ in the original equation that we have the right answer!

-3- **(A Quadratic Equation.)** Find all solutions of the equation

$$x^2 - x - 20 = 0$$

This equation can be solved factoring, by the quadratic formula, or by completing the square. Factoring gives

$$x^2 - x - 20 = (x - 5)(x + 4) \tag{2}$$

which implies that the solution is

$$x = -4 \quad \text{or} \quad x = 5 \tag{3}$$

Completing the square gives:

$$
\begin{array}{rcll}
x^2 - x - 20 & = & 0 & +20 + \frac{1}{4} \\
x^2 - x + \frac{1}{4} & = & \frac{81}{4} & \text{perfect square} \\
\left(x - \frac{1}{2}\right)^2 & = & \frac{81}{4} & \sqrt{} \\
x - \frac{1}{2} & = & \pm\frac{9}{2} & +\frac{1}{2} \\
x & = & \frac{1}{2} \pm \frac{9}{2} & \text{the solution}
\end{array}
\tag{4}
$$

which is the same result.
As for all quadratic equations, you can also apply the quadratic formula if you can remember it reliably.

-4- **(Another Quadratic Equation.)** Find all solutions of the equation

$$x^2 - 2x - 5 = 0$$

Proceeding similarly as in the preceding problem we obtain:

$$
\begin{array}{rcll}
x^2 - 2x - 5 & = & 0 & +6 \\
x^2 - 2x + 1 & = & 6 & \text{perfect square} \\
(x - 1)^2 & = & 6 & \sqrt{} \\
x - 1 & = & \pm\sqrt{6} & +1 \\
x & = & 1 \pm \sqrt{6} & \text{the solution}
\end{array}
\tag{5}
$$

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-5- **(A rational equation.)** Find all solutions of

$$\frac{8}{x-2} - \frac{5}{x-3} + 1 = 0$$

This equation can be converted into a quadratic equation by multiplying on both sides of the equation by the common denominator $(x-2)(x-3)$. We obtain:

$$8(x-3) - 5(x-2) + (x-2)(x-3) = 3x - 14 + x^2 - 5x + 6 = x^2 - 2x - 8 = 0. \qquad (6)$$

The last expression can be factored:

$$x^2 - 2x - 8 = (x+2)(x-4) \qquad (7)$$

Thus

$$x = -2 \quad \text{or} \quad x = 4. \qquad (8)$$

Substituting in the original equation shows that this is indeed the correct solution.

-6- **(Polynomials.)** Write the following polynomial expression in standard form. What is its degree and its leading coefficient?

$$(x^2 - 1)(x+3) + 2x + 4$$

We obtain

$$(x^2 - 1)(x+3) + 2x + 4 = x^3 + 3x^2 - x - 3 + 2x + 4 = x^3 + 3x^2 + x + 1.$$

This is a polynomial expression of degree 1, with leading coefficient 1.

For the next three questions let

$$f(x) = \frac{x-1}{x^2 - 2x}$$

-7- **(Domain.)** What is the natural domain of f? It's the set of a real numbers except 0 and 2, since we must not divide by zero.

-8- **(Evaluating at a number.)** Find

$$f(5) = \frac{5-1}{5^2 - 10} = \frac{4}{15}.$$

-9- **(Evaluating at an expression.)** Find $f(2x+1)$ and it express it in the standard form of a rational expression.
We obtain

$$f(2x+1) = \frac{2x+1-1}{(2x+1)^2 - 2(2x+1)} = \frac{2x}{4x^2 + 4x + 1 - 4x - 2} = \frac{2x}{4x^2 - 1}.$$

-10- **(Complex Numbers.)** Calculate $\frac{1+i}{1+4i}$. (Here $i = \sqrt{-1}$. Your answer should be in the form $a + bi$ where a and b are real numbers.)
We multiply numerator and denominator with the conjugate complex of the denominator, and obtain:

$$\frac{1+i}{1+4i} = \frac{(1+i)(1-4i)}{(1+4i)(1-4i)} = \frac{1 - 4i + i - 4i^2}{1 + 4^2} = \frac{5 - 3i}{17} = \frac{5}{17} - \frac{3}{17}i.$$

-11- **(Linear System.)** Solve the system

$$\begin{array}{rcl} 4x \quad -y &=& 1 \\ 2x \quad +y &=& 0 \end{array}$$

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State how you solve it, don't just give the answer.
Adding the two equations gives $6x = 1$ which implies

$$x = \frac{1}{6}.$$

Substituting in the first equation gives $\frac{4}{6} - y = 1$ which implies

$$y = \frac{4}{6} - 1 = -\frac{1}{3}.$$

Substituting these values of x and y in the original equations shows that we have the right solution.

-12- (Another Linear System.) Solve the linear system

$$
\begin{array}{rrrcr}
x & +y & +z & = & 5 \\
x & +2y & -z & = & 3 \\
2x & +3y & -z & = & 5
\end{array}
$$

Again, state how you solve it, don't just give the answer.
This is a little more elaborate. We write equation numbers in square brackets and obtain:

$$
\begin{array}{rrrcrl}
x & +y & +z & = & 5 & \quad [1] \\
x & +2y & -z & = & 3 & \quad [2] \\
2x & +3y & -z & = & 5 & \quad [3] \\
 & y & -2z & = & -2 & \quad [4] = [2] - [1] \\
 & y & -3z & = & -5 & \quad [5] = [3] - 2 \times [1] \\
 & & -z & = & -3 & \quad [6] = [4] - [5]
\end{array}
\tag{9}
$$

Thus $z = 3$, and we obtain from equation [4] that $y = 2z - 2 = 4$. Finally, we obtain from equation [1] that $x = 5 - y - z = -2$. Thus

$$x = -2, \qquad y = 4, \qquad z = 3. \tag{10}$$

-13- (Straight Lines.) Find an equation of the line that passes through (2,1) and has slope -1/2. Draw its graph.
The point slope form of the equation of a straight line gives

$$\frac{y-1}{x-2} = -\frac{1}{2}.$$

This can be rewritten as, for example,
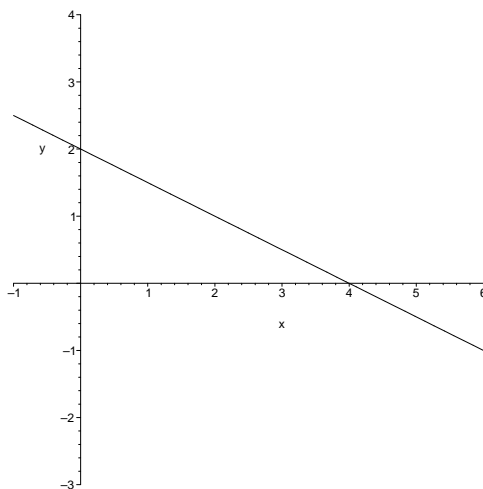
$$y = 1 - \frac{1}{2}(x - 2) = -\frac{1}{2}x + 2.$$

Figure 1. Graph of $y = -\frac{x}{2} + 2$, question 14.

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The graph is shown in Figure 1.

-14- **(Distance.)** Find the distance between the points $(-1, 3)$ and $(2, 4)$.
We apply the distance formula, giving:

$$d = \sqrt{(-1-2)^2 + (3-4)^2} = \sqrt{10}.$$

-15- **(Powers.)** Simplify (i.e., write with only positive exponents, such that x and y occur only once) the expression

$$\frac{(x^2 y^{-3})^2}{(x^{-1} y)^{-3}}$$

We apply the Rules

$$a^m a^n = a^{m+n}, \qquad (a^m)^n = a^{mn}, \qquad (ab)^n = a^n b^n, \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

This gives

$$\frac{\left(x^2 y^{-3}\right)^2}{\left(x^{-1} y\right)^{-3}} = \frac{x^4 y^{-6}}{x^3 y^{-3}} = x^{4-3} y^{-6+3} = xy^{-3} = \frac{x}{y^3}.$$

-16- **(Radical Expressions.)** Simplify the expression

$$\left(\frac{x^{3/2}}{x^{\frac{1}{6}}}\right)^{2/5}$$

and write it as a power with a single exponent.
Proceeding similarly, we obtain

$$\left(\frac{x^{3/2}}{x^{\frac{1}{6}}}\right)^{2/5} = \left(x^{3/2-1/6}\right)^{2/5} = x^{2/5(3/2-1/6)} = x^{8/15}.$$

-17- **(Rational Expressions.)** Write $\frac{2}{x+1} + \frac{1}{x-2} + \frac{2}{x-3}$ in the standard form of a rational expression.
The common denominator of these three expressions is just the product of the individual denominators. Thus we obtain

$$\frac{2}{x+1} + \frac{1}{x-2} + \frac{2}{x-3} = \frac{2(x-2)(x-3) + (x+1)(x-3) + 2(x+1)(x-2)}{(x+1)(x-2)(x-3)}$$

$$= \frac{2(x^2-5x+6) + (x^2-2x-3) + 2(x^2-x-2)}{(x+1)(x-2)(x-3)}$$

$$= \frac{5x^2 - 14x + 5}{(x+1)(x-2)(x-3)}.$$

-18- **(A Word Problem.)** Twenty-one years ago Jack was twice as old as Jill. Fourteen years ago Jack was only one and a half times as old as Jill. How old are Jack and Jill today?
You can find th answer by setting up and solving a linear system of equations. Letting x denote the current age of Jack, and y the current age of Jill, the equations are

$$x - 21 = 2(y - 21) \quad \text{and} \quad x - 14 = \frac{3}{2}(y - 14).$$

This becomes, after rearranging,

$$x - 2y = -21$$

$$x - \frac{3}{2}y = -7$$

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Subtracting the first equation from the second gives

$$\frac{1}{2}y = 14$$

Thus $y = 28$. Substituting in the first equation gives $x - 56 = -21$ and hence $x = 35$.

Jack is 35 and Jill is 28.

21 years ago Jack was 14, and Jill was 7. 14 years ago, Jack was 21 and Jill was 14. It all fits!
If you have younger siblings, and you have watched your seniority shrink relative to theirs, the following might have occurred to you: When a new sibling is born, you are infinitely as old as she is. By the time your age at the birth has passed you are only twice as old as your sibling. Another period of the same length later, you are only one and a half times as old as she is. She's catching up! Since that key period in the problem is 7 years, when Jack was twice as old as Jill, he must have been 14, and Jill 7. Since this was 21 years ago, now they must be 35 and 28. The author of this problem is the oldest of six children, and as he grew up he did this sort of calculation frequently, usually in lieu of counting sheep before falling asleep.

-19- **(Another Word Problem.)** You and your brother working together take 3 hours to dig a hole. By yourself it would take you five hours. How long would it take your brother to dig the hole by himself?
In one hour you dig $1/5$ of the hole, the two of you dig $1/3$ of the hole, and your brother digs $1/x$ of the hole, where x is the number of hours it takes him by himself.
We need to solve the linear equation

$$\frac{1}{x} + \frac{1}{5} = \frac{1}{3}.$$

This means

$$\frac{1}{x} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

and hence

$$x = \frac{15}{2}.$$

It takes your brother seven and a half hours by himself! He is still young!

-20- **(Yet Another Word Problem.)** You obtain 5 gallons of 14% vinegar by mixing suitable amounts of 5% vinegar and of 20% vinegar. How much of each type of vinegar do you use?
Let's suppose we use x gallons of 5% vinegar (and hence $5 - x$ gallons of 20 % vinegar). The acidity (amount of acid divided by amount of fluid) of the resulting fluid is

$$\frac{0.05x + 0.2(5 - x)}{5}. \tag{11}$$

We want it to equal 0.14:

$$\frac{0.05x + 0.2(5 - x)}{5} = 0.14. \tag{12}$$

Multiplying with 5 and rearranging the the left side gives

$$-0.15x + 1 = 0.7 \tag{13}$$

or

$$-0.15x = -0.3 \tag{14}$$

Thus

$$x = 2. \tag{15}$$

We use 2 gallons of 5% vinegar and 3 gallons of 20% vinegar. This makes sense. A mixture of equal amounts of both would have an acidity of 12.5%, and using more of the stronger vinegar makes the mixture stronger also.