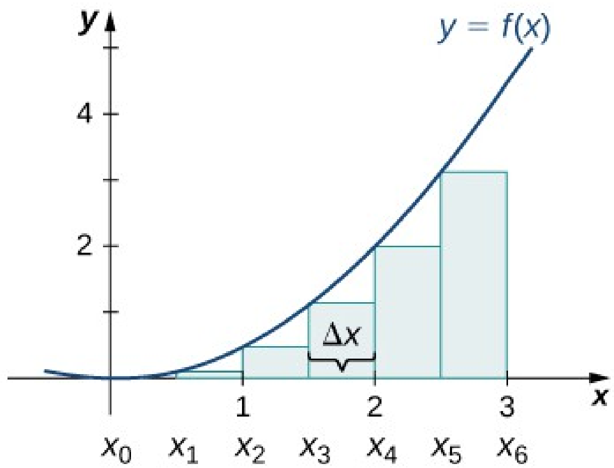
13.1 Area Under a Curve

Ex 1: Write in sigma (summation) notation. (a) $a_1 - a_2 + a_3 - a_4 + a_5 - \ldots$ (b) $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \ldots$	Summation notation: $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$ left side sigma notation is read "the sum from i equals 1 to n of a sub i" Special Sum Formulas: 1. $\sum_{i=1}^{n} 1 = n$ and $\sum_{i=1}^{n} c = cn$ where $c \in \mathbb{R}$ 2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ 3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ 4. $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$
Ex 2: Find the sum (using the special sum formula(s)) $\sum_{k=1}^{20} (2k^2 - 3)$	Finite summation is a linear operator: This means, once again, it satisfies two conditions, namely (a) it distributes through addition, i.e. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ AND (b) it commutes with scalar multiplication (or multiplication by a constant), i.e. $\sum_{i=1}^{n} c\,a_i = c\sum_{i=1}^{n} a_i$ for a constant $c \in \mathbb{R}$.

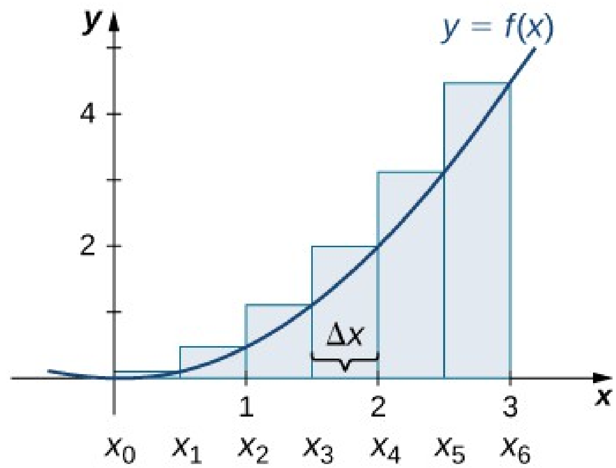
Why do we care about summation? Because the area under a curve, in 2d, can be approximated by the sum of areas of rectangles "under" the curve.

left-endpoint subintervals



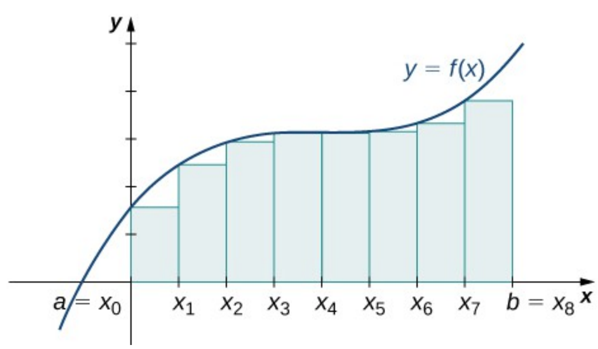
(a) This shows rectangles whose area approximates the area under the curve of y=f(x) between x=0 to x=3, where all rectangles have heights chosen by the left x-value (endpoint) of the subdivided intervals.

right-endpoint subintervals



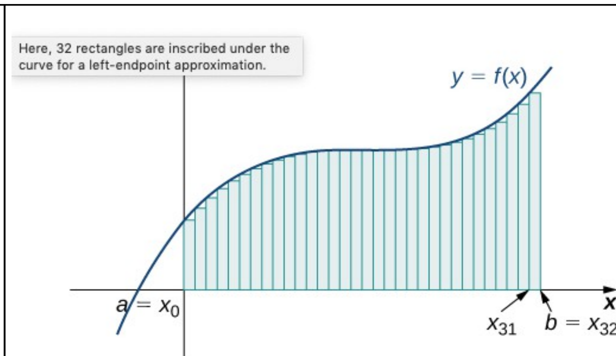
(b) This shows rectangles whose area approximates the area under the same curve between x=0 and x=3, where all rectangles have heights chosen by the right x-value (endpoint) of the subdivided intervals.

And, if we keep increasing the number of subintervals that we divide up the overall x-interval into, then we can get a closer and closer approximation for the actual area between the curve and the x-axis.



This x-interval from x=a to x=b is subdivided into 8 equal sized subintervals.
The sum of the areas of these rectangles estimates the area under the curve.

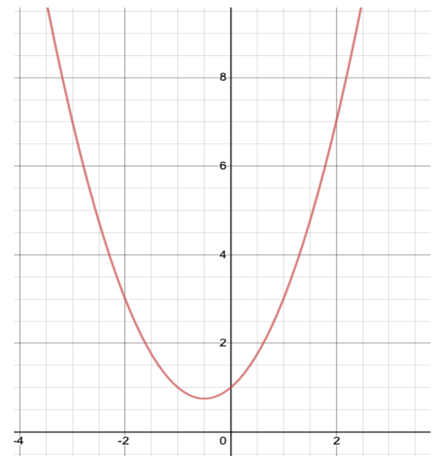
Question: Is this picture showing you left or right endpoints?



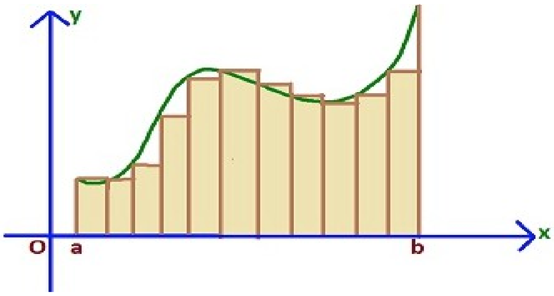
This x-interval from x=a to x=b is subdivided into 32 equal sized subintervals.
The sum of the areas of these rectangles more closely estimates the area under the curve.

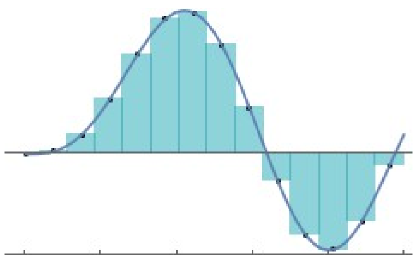
More compelling question: Is there some way to get the area under the curve exactly? Or do we have to keep adding more and more narrow-widthed rectangle areas together to get the approximate area?

Ex 3: Approximate the area under the curve $y = f(x) = x^2 + x + 1$ from x=-1 to x=1, using both left end points and right endpoints, with four sub-intervals.



13.2 The Definite Integral: The Fundamental Theorem of Calculus (FTC)





You see in these pictures that some rectangles are "too big" and other rectangles are "too small", but the sum of hte areas of those rectangles will give an approximate value for the actual area between the curve and the x-axis. And if we keep making those rectangles skinnier and skinnier in width, then eventually the area will be exact.

<u>(One) Definition of the definite integral</u>:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

If we choose right-hand x-values for each rectangle, of uniform width, then we have the following formulas.

$$\Delta x = \frac{b-a}{n}$$

and

$$x_i = a + i(\Delta x)$$

(*This is the "long" way to calculate a definite integral. We won't really do this in practice.*)

<u>Fundamental Theorem of Calculus</u>:

If $f(x)$ is continuous on [*a, b*], and $F(x)$ is any antiderivative, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad .$$

(*This is the "short-cut" way to evaluate definite integrals, and this is what we'll use in practice.*)

Geometrically, this represents the area between the function y=f(x) and the x-axis.

More properties of definite integrals:

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

The definite integral is also a linear operator. You list the two properties it must meet then.

(a) $\int_a^b (f(x)+g(x))\,dx$

= ____

AND

(b) $\int_a^b k\,f(x)\,dx$ = __________________________

for any constant k.

13.2 (continued)

Ex 1: Using the FTC, calculate these definite integrals.

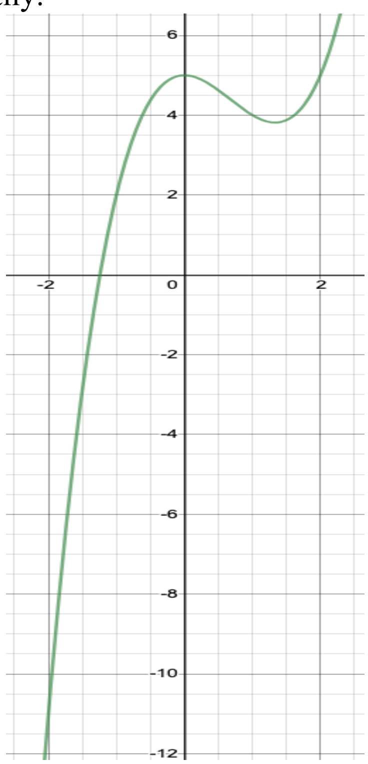
(a) $\int_{0}^{3} (4x)\,dx$	(b) $\int_{-1}^{2} (3x^2+4)\,dx$
(c) $\int_{-8}^{0} \frac{3}{\sqrt{9-2x}}\,dx$	(d) $\int_{-1}^{0} 30(x+3)(x+1)^4\,dx$

13.2 (continued)

Ex 2: Using the FTC, calculate the integral $\int_{0}^{\ln 5} (10e^{x} - 20e^{-x})\,dx$.

Ex 3: For this curve, $y = x^3 - 2x^2 + 5$ (graph of this curve is given below), answer the following questions.

(a) Shade in what $\int_{-2}^{2} (x^3 - 2x^2 + 5)\,dx$ means geometrically.

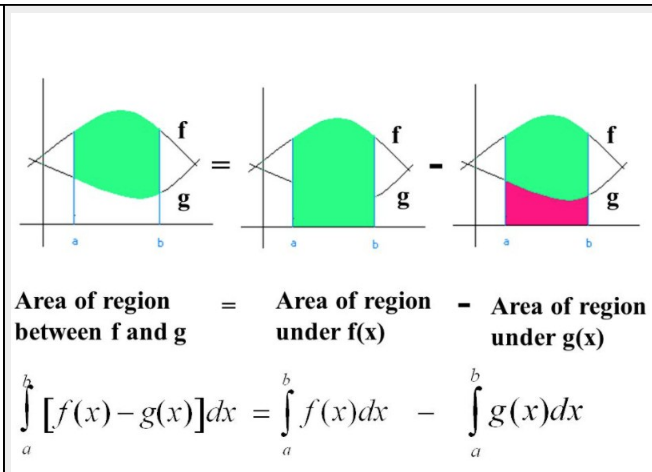


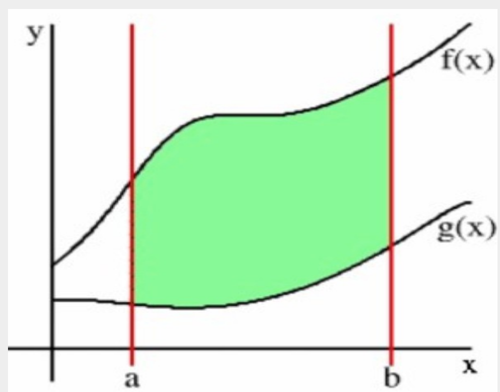
(b) Use FTC to evaluate the integral given in (a).

13.3 Area Between Two Curves

Ex 1: Find the area between these curves.

$y=\sqrt{x}$, $y=x-4$, $x=0$





Area between curves = $\int_a^b (f(x)-g(x))\,dx$

Ex 2: Find the area between the two curves $y=x$ and $y=6-x^2$.

Ex 3: Find the area between the curves $y=\frac{1}{x^2}$, $y=x$ and $y=\frac{1}{8}x$.

Ex 4: Find the average value of the function $f(x)=2x-x^2$ on the interval $[0,2]$.

Average Function Value:

If $f(x)$ is continuous on [a, b], then the average value of f is given by

$$\text{average function value}=f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx \quad .$$

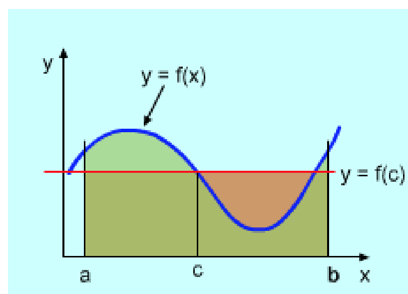
f(c) is called the average value of the function on that interval.
And, c is the x-value where that average value occurs.

Ex 5: I bet you're wondering how Hot Chippies is doing. Thanks for asking. It's a great business so far. My mom has hired an assistant now. They have worked out that the cost of baking x cookies is given by

$C(x)=x^2+400x+2000$.

(a) Use this cost function to figure out the average cost of baking 1000 cookies.

(b) Find the average value of the cost function over the interval from 0 to 1000 cookies.



13.4 Applications of Definite Integrals in Business and Economics

Ex 1: Find the total income over the next 8 years from a continuous income stream with an annual rate of flow at time t given by $f(t)=8500e^{-0.1t}$ (dollars per year).	<u>Continuous Income Stream:</u> Suppose f(t) is the annual rate of flow of income for some business, where t represents time in years. Then we can find the total income using integration. The **total income for k years** is given by $\int_0^k f(t)dt$. Furthermore, if f(t) earns interest at rate r, compounded continuously, then the **present value** (PV) of the continuous income stream over those k years is $PV=\int_0^k f(t)e^{-rt}dt$. And the f**uture value** (FV) of the continuous income stream (earning annual interest at rate r compounded continuously over k years) is $FV=e^{rk}\int_0^k f(t)e^{-rt}dt$
	Note: Remember from Math1090 (Business Algebra) that, in general, PV is the current lump sum value of some investment that will produce payments in the future. And FV is the lump sum future value that will be produced by putting payments in between now and the future date.

Ex 2: My mom is so excited about Hot Chippies that she thinks it would be a good idea to expand and purchase another small business that aligns with making and selling cookies. She's found a small brownie shop that she might purchase, with proceeds from our Hot Chippies. The brownie shop has a continuous income stream with an annual rate of flow at time t given by $B(t)=15{,}000\,e^{0.06t}$ (dollars per year). The money is worth 8% compounded continuously. Find the PV of the business over the next 6 years.

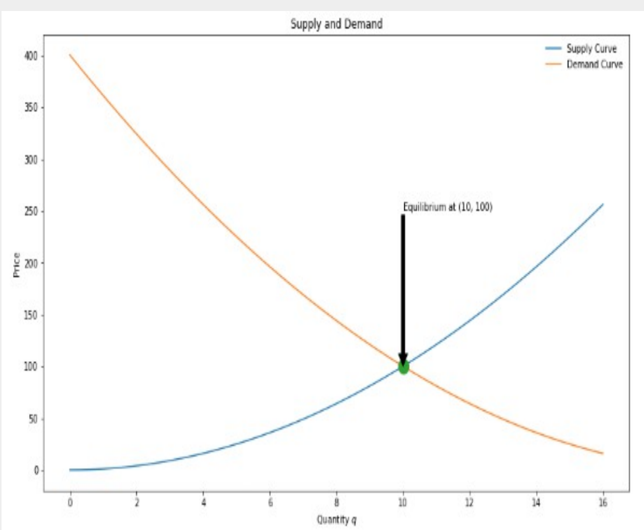
Ex 3: The demand function for a product is $p = 100 - 4x$ and the equilibrium price is \$40. What is the consumer's surplus for this product?

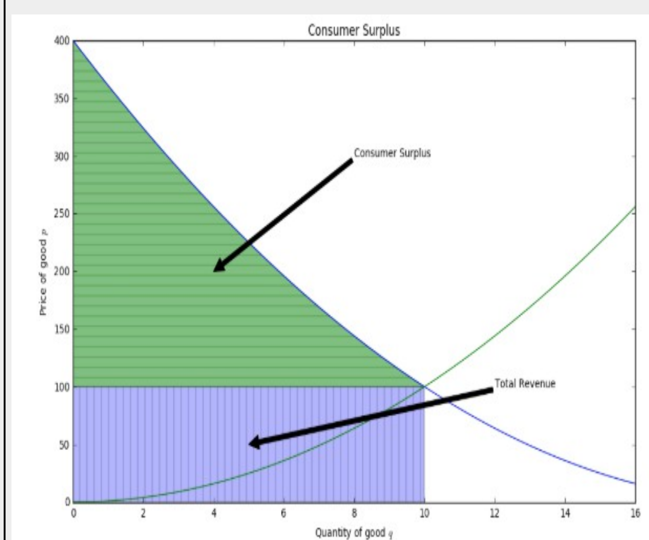
Consumer's Surplus:

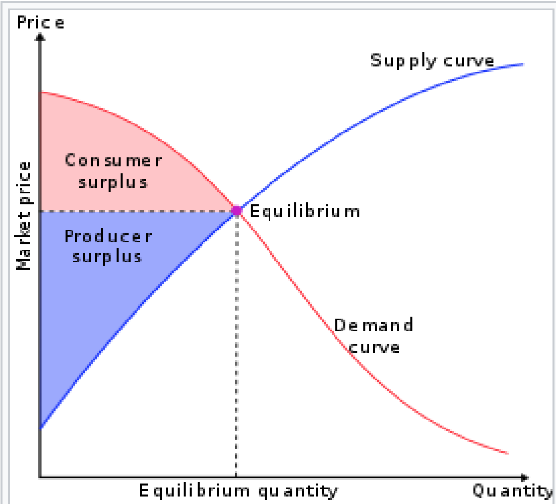
Suppose demand for a product is given by p=f(x) and supply of the product is described by p=g(x). Then the price, p_1 , where the graphs of these functions intersect is the **equilibrium point** (which gives the equilibrium price and quantity where supply and demand match).

Some consumers are willing to pay more than the equilibrium price, which means they benefit from purchasing the item at the equilibrium price. The total gain for all those consumers willing to pay more than $\$ p_1$ is called the **consumer's surplus** (CS) and is given by

$$CS = \left(\int_0^{x_1} f(x)\,dx \right) - p_1 x_1 \quad .$$





Ex 4: If the supply function for a product is $p=4x+4$ and the demand function is $p=49-x^2$, find (a) the equilibrium point and (b) the PS.	Producer's Surplus: If the supply function is given by p=g(x), the **producer's surplus** (PS) is given by the area between the graph of p=g(x) and the x-axis from 0 to x_1 subtracted from the area of the rectangle that has the origin on one corner and the equilibrium point on the diagonally opposite corner. That is, $PS=p_1x_1-\left(\int_0^{x_1} g(x)\,dx\right)$  Graph illustrating consumer (red) and producer (blue) surpluses on a supply and demand chart

13.6 Integration by Parts

Ex 1: Compute these integrals. (a) $\int \sqrt{2x}\ln(x)\,dx$	<u>Formula for doing Integration by Parts</u>: $\int u\,dv = uv - \int v\,du$ Note: This formula is based on the "undoing" of the product rule for differentiation.
(b) $\int_0^1 x^2 e^x\,dx$	Class cases of integrals that need integration by parts technique: An integral where the integrand function is 1. the product of a polynomial and an exponential function. 2. One single function that is straightforward to differentiate but we don't immediately know how to integrate it. 3. The product of x to a power times a logarithmic function.

13.6 (continued)

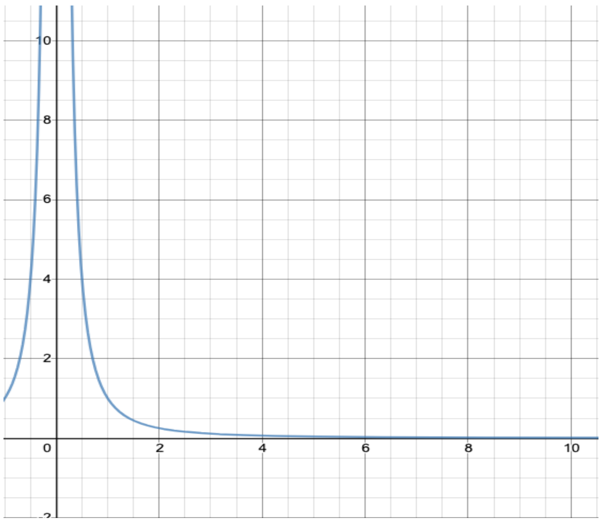
Ex 2: Compute the following integrals.

(a) $\int \ln(5x)\,dx$

(c) $\int_0^1 x^3 \sqrt{x^2+4}\,dx$

13.7 Improper Integrals and Their Applications

Ex 1: Evaluate the integral.



$$\int_{1}^{\infty} \frac{1}{x^2}\,dx$$

Remember that this is asking for the area between the curve and the x-axis as x goes from 1 to positive infinity.

Improper Integrals:
Essentially these are like definite integrals except that one or both integration bounds are some sort of infinity (positive or negative). We're asking for the area under the curve where the curve goes on forever to the right and/or left.

Strangely enough, this area CAN be finite, even though x is going off to infinity! It's remarkable.

You must first write the integral as a limit and then proceed as if it's a regular definite integral. At the very end of that process, you take the limit and see what you get.

(1) $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx$

(2) $\int_{a}^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_{a}^{b} f(x)\,dx$

(3) $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{0} f(x)\,dx + \lim_{b\to\infty} \int_{0}^{b} f(x)\,dx$

Note: If these limits, for (1) and (2) cases, evaluate to be a finite value, then the integral is said to converge and the answer is that finite value.

If these limits, for (1) and (2) cases, either do not exist or go to some sort of infinity, then the integral is said to diverge.

For case (3), if one of those integrals (on the right) diverges, then the entire integral diverges. That is, (3) converges to a finite value only if both pieces converge.

What's necessary for an improper integral to possibly converge?

13.7 (continued)

Logic Note: Let's take a look at what we mean by necessary vs. sufficient conditions, and we'll use an analogy to help us think through the logic of this.

1. If a dog is pregnant, then we can conclude that the dog is female. So the dog being pregnant is sufficient to conclude it's female.	2. If a dog is female, then we know there's a chance the dog is pregnant. So the dog being female is necessary before we even ask whether or not it's pregnant.
Here's what we have logically: 1. pregnant dog ==> female dog 3. male (i.e. not female) dog ==> not pregnant 2. female dog implies nothing about pregnancy except it's worth checking.	3. On the other hand, if we know that the dog is male, then we're not going to ask the question if the dog is pregnant.

How does this compare or relate to improper integrals?

1. If the improper integral converges to a finite number (i.e. the area under the curve is finite), then we can conclude that the integrand function goes to zero as x goes to (plus or minus) infinity. So the convergence of the integral is sufficient to conclude the integrand function has a HA of y=0.	2. If the integrand function of an improper integral has a limit of zero as x goes to (positive or negative, whichever is asked for) infinity, then we know there's a chance that the improper integral converges. So the integrand going off to zero (having a HA of y=0) is necessary before we even ask whether or not the integral converges. (*If the integrand function goes to zero "fast enough", whatever that means, then the integral will converge.*)
Here's what we have logically: 1. convergent improper integral ==> integration function eventually has a y-value that's closer and closer to zero as x gets huge 3. the integrand function does NOT have HA of y=0 ==> integral diverges 2. integrand function has HA of y=0 implies nothing about convergence of integral; it's worth doing the integral to see what happens.	3. On the other hand, if we know that the integrand function does NOT have a HA of y=0, then we're not going to ask the question if the integral converges. In this case, we already know the answer: the integral diverges.

13.7 (continued)

Ex 2: Evaluate the following integrals.

(a) $\int_{9}^{\infty} \frac{3}{x+1}\,dx$	(b) $\int_{-\infty}^{-1} \frac{1}{x^3}\,dx$
(c) $\int_{-\infty}^{\infty} \frac{2x}{\sqrt{x^2+25}}\,dx$	(d) $\int_{-\infty}^{\infty} \frac{x\,e^{-x^2/2}}{\sqrt{2\pi}}\,dx$