Chapter 6

A. Trigonometry Review (Inverse functions and their ranges)

$x=\sin^{-1}y \Leftrightarrow y=\sin x$, $x\in[\frac{-\pi}{2},\frac{\pi}{2}]$

$x=\cos^{-1}y \Leftrightarrow y=\cos x$, $x\in[0,\pi]$

$x=\tan^{-1}y \Leftrightarrow y=\tan x$, $x\in(\frac{-\pi}{2},\frac{\pi}{2})$

$x=\sec^{-1}y \Leftrightarrow y=\sec x$, $x\in[0,\pi]$, $x\neq\frac{\pi}{2}$

B. More Trigonometry Review (Equivalent inverse functions)

$\sec^{-1}y=\cos^{-1}\left(\frac{1}{y}\right)$

$\cot^{-1}y=\tan^{-1}\left(\frac{1}{y}\right)$

$\csc^{-1}y=\sin^{-1}\left(\frac{1}{y}\right)$

C. Equivalent Algebraic and Trigonometric Expressions

$\sin(\cos^{-1}(x))=\sqrt{1-x^2}=\cos(\sin^{-1}(x))$

$\sec(\tan^{-1}(x))=\sqrt{1+x^2}$

$\tan(\sec^{-1}(x))=\begin{cases}\sqrt{x^2-1}, & x\geq 1\\ -\sqrt{x^2-1}, & x\leq -1\end{cases}$

D. Calculus Review (Trigonometric Derivatives)

$D_x(\sin x)=\cos x$

$D_x(\cos x)=-\sin x$

$D_x(\tan x)=\sec^2 x$

$D_x(\sec x)=\sec x\tan x$

$D_x(\csc x)=-\csc x\cot x$

$D_x(\cot x)=-\csc^2 x$

A. Hyperbolic Identities/Definitions

$\cosh^2 x-\sinh^2 x=1$

$\sinh x=\frac{1}{2}(e^x-e^{-x})$

$\cosh x=\frac{1}{2}(e^x+e^{-x})$

B. Derivatives of Hyperbolic Functions

$D_x(\sinh x)=\cosh x$

$D_x(\cosh x)=\sinh x$

$D_x(\tanh x)=\text{sech}^2 x$

$D_x(\text{sech}\, x)=-\text{sech}\, x\tanh x$

$D_x(\coth x)=-\text{csch}^2 x$

$D_x(\text{csch}\, x)=-\text{csch}\, x\coth x$

C. Inverse Hyperbolic Function Algebraic Equivalent Statements

$\sinh^{-1}x=\ln(x+\sqrt{x^2+1})$

$\cosh^{-1}x=\ln(x+\sqrt{x^2-1})$, $x\geq 1$

$\tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $x\in(-1,1)$

$\text{sech}^{-1}x=\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$, $x\in(0,1]$

D. Derivatives of Inverse Hyperbolic Functions

$D_x(\sinh^{-1}x)=\frac{1}{\sqrt{x^2+1}}$

$D_x(\cosh^{-1}x)=\frac{1}{\sqrt{x^2-1}}$, $x>1$

$D_x(\tanh^{-1}x)=\frac{1}{1-x^2}$, $x\in(-1,1)$

$D_x(\text{sech}^{-1}x)=\frac{-1}{x\sqrt{1-x^2}}$, $x\in(0,1)$

E. New: Derivatives of Inverse Trigonometric Functions

$D_x(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$, $x\in(-1,1)$

$D_x(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$, $x\in(-1,1)$

$D_x(\tan^{-1}x)=\frac{1}{1+x^2}$

$D_x(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$, $|x|>1$

$D_x(a^x)=a^x(\ln a)$, $a>0$

$\int a^x\,dx=\frac{a^x}{\ln a}+C$, $a>0$, $a\neq 1$

$D_x(\log_a x)=\frac{1}{x\ln a}$

$\int x^a\,dx=\frac{x^{a+1}}{a+1}+C \quad \forall a\neq -1$

$\frac{dy}{dt}=ky \Leftarrow y=y_0e^{kt}$

If $k>0$, it's exponential growth

If $k<0$, it's exponential decay

Chapter 7

A. Form $\int\sin^n x\,dx$ or $\int\cos^n x\,dx$:

If n is odd, use the Pythagorean identity $\sin^2 x+\cos^2 x=1$.

If n is even, use the half-angle identities $\sin^2 x=\frac{1-\cos 2x}{2}$ and $\cos^2 x=\frac{1+\cos 2x}{2}$.

B. Form $\int\sin^m x\,\cos^n x\,dx$:

If m or n is odd, use Pythagorean identity.

If both m and n are even, use half-angle identities.

C. Form $\int\sin(mx)\cos(nx)dx$ or $\int\sin(mx)\sin(nx)dx$ or $\int\cos(mx)\cos(nx)dx$:

Use product identities.

$\sin(mx)\cos(nx)=\frac{1}{2}(\sin((m+n)x)+\sin((m-n)x))$

$\sin(mx)\sin(nx)=\frac{-1}{2}(\cos((m+n)x)-\cos((m-n)x))$

$\cos(mx)\cos(nx)=\frac{1}{2}(\cos((m+n)x)+\cos((m-n)x))$

A. Form $\sqrt[n]{ax+b}$:

Try u-sub with $u=\sqrt[n]{ax+b}$.

Important note: This is the root of a LINEAR polynomial, i.e. power on x is 1.

B. Form $\sqrt{\text{quadratic polynomial}}$:

(a) For $\sqrt{a^2-x^2}$ let $x=a\sin\theta$, $\theta\in[\frac{-\pi}{2},\frac{\pi}{2}]$

(b) For $\sqrt{a^2+x^2}$ let $x=a\tan\theta$, $\theta\in(\frac{-\pi}{2},\frac{\pi}{2})$

(c) For $\sqrt{x^2-a^2}$ let $x=a\sec\theta$, $\theta\in[0,\pi]$, $\theta\neq\frac{\pi}{2}$

C. Extra note of useful integrals to have on your note card:

$\int\csc x\,dx=\ln|\csc x-\cot x|+C$

$\int\sec x\,dx=\ln|\sec x+\tan x|+C$

(1) $\int_{-\infty}^{b}f(x)dx=\lim_{a\to-\infty}\int_a^b f(x)dx$

(2) $\int_a^{\infty}f(x)dx=\lim_{b\to\infty}\int_a^b f(x)dx$

(3) $\int_{-\infty}^{\infty}f(x)dx=\lim_{a\to-\infty}\int_a^0 f(x)dx+\lim_{b\to\infty}\int_0^b f(x)dx$

Note: If these limits, for (1) and (2) cases, evaluate to be a finite value, then the integral is said to converge and the answer is that finite value.

If these limits, for (1) and (2) cases, either do not exist or go to some sort of infinity, then the integral is said to diverge.

For case (3), if one of those integrals (on the right) diverges, then the entire integral diverges. That is, (3) converges to a finite value only if both pieces converge.

(1) If there is a VA (vertical asymptote) at x = b, i.e. $\lim_{x\to b^-}f(x)=\pm\infty$, and $f(x)$ is continuous on [a,b), then $\int_a^b f(x)dx=\lim_{m\to b^-}\int_a^m f(x)dx$.

If this limit goes to infinity or DNE, then the integral diverges.

(2) If there is a VA (vertical asymptote) at x = a, i.e. $\lim_{x\to a^+}f(x)=\pm\infty$, and $f(x)$ is continuous on (a,b], then $\int_a^b f(x)dx=\lim_{m\to a^+}\int_m^b f(x)dx$.

If this limit goes to infinity or DNE, then the integral diverges.

<u>Non-indeterminate Forms:</u>

(there's no competition going on here)

$\frac{0}{\infty}\to 0$

$\frac{\infty}{0}\to\infty$

$\infty+\infty\to\infty$

$\infty\cdot\infty\to\infty$

$0^\infty\to 0$

$\infty^\infty\to\infty$

$1^0\to 1$

(3) If there is a VA (vertical asymptote) at x = c, and $c\in(a,b)$ then $\int_a^b f(x)dx=\lim_{m\to c^-}\int_a^m f(x)dx+\lim_{p\to c^+}\int_p^b f(x)dx$.

Chapter 8

<u>L'Hopital's Rule</u>

If $\lim_{x\to u}f(x)=0$ and $\lim_{x\to u}g(x)=0$, then $\lim_{x\to u}\frac{f(x)}{g(x)}=\lim_{x\to u}\frac{f'(x)}{g'(x)}$.

<u>L'Hopital's Rule</u>

If $\lim_{x\to u}f(x)=\pm\infty$ and $\lim_{x\to u}g(x)=\pm\infty$, then $\lim_{x\to u}\frac{f(x)}{g(x)}=\lim_{x\to u}\frac{f'(x)}{g'(x)}$.

<u>Indeterminate Limit Forms:</u>

$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞ cases

All of these cases are "competing."

Chapter 9

A sequence $\{a_n\}$ converges if $\lim_{n\to\infty}a_n=$ finite number. Otherwise, $\{a_n\}$ diverges.

(In other words, the limit could be some sort of infinity or the limit could not exist and the sequence would diverge.)

Assuming $\sum_{n=1}^{\infty}a_n$ is a positive series (meaning that each of the a_n terms are positive), you can use these tests to determine convergence or divergence of the series.

(1) <u>nth term test for DIVERGENCE</u>:

If $\lim_{n\to\infty}a_n\neq 0$, then $\sum_{n=1}^{\infty}a_n$ diverges.

(2) <u>Geometric Series</u>:

For series of the form $\sum_{n=k}^{\infty}ar^n$ where k and a are constants,

$\sum_{n=k}^{\infty}ar^n=\frac{\text{first term}}{1-r}$ if $|r|<1$. Otherwise

$\sum_{n=k}^{\infty}ar^n$ diverges if $|r|\geq 1$.

(And, first term = the term in the series when you plug in the first value of n, i.e. ar^k.)

Put on your notecard:

$\lim_{h\to 0}(1+h)^{\frac{1}{h}}=e$

and equivalently $\lim_{x\to\infty}(1+\frac{1}{x})^x=e$

Positive Series Tests:

Assuming $\sum_{n=1}^{\infty}a_n$ is a positive series.

(1) <u>nth term test for DIVERGENCE</u>:

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$\sum_{n=k}^{\infty}ar^n$ diverges if $|r|\geq 1$.

(3) <u>p-series</u>:

$\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}\text{converges, if } p>1\\ \text{diverges, if } p\leq 1\end{cases}$

(7) <u>Integral Test</u>:

If f is a (a) continuous, (b) positive, and (c) non-increasing function on $[k,\infty)$, then $\sum_{n=k}^{\infty}a_n$ converges iff $\int_k^{\infty}f(x)dx$ where $a_n=f(n)$.

(5) <u>RT (Ratio Test)</u>:

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\rho\begin{cases}\text{if } \rho<1\text{, the series converges}\\ \text{if } \rho>1\text{, the series diverges}\\ \text{if } \rho=1\text{, there's no conclusion}\end{cases}$

(If there is no conclusion, it means you have to try a different test until you get a conclusion.)

(6) <u>OCT (Ordinary Comparison Test)</u>:

If $0\leq a_n\leq b_n$ for $n\geq N$ (for some finite N value), then:

$\sum_{n=1}^{\infty}b_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty}a_n$ converges

$\sum_{n=1}^{\infty}a_n$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty}b_n$ diverges

Note: This is the order of tests I prefer. The other tests are given on a previous page, from the last section of notes.

Positive Series Tests:

Assuming $\sum_{n=1}^{\infty}a_n$ is a positive series.

(4) <u>LCT (Limit Comparison Test)</u>:

If $a_n\geq 0$, $b_n\geq 0$ and $\lim_{n\to\infty}\frac{a_n}{b_n}=L$ and if $0<L<\infty$, then $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ converge or diverge together.

If L = 0 AND $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Otherwise, there is no conclusion.

(Note: $\sum_{n=1}^{\infty}a_n$ refers to the series that we are given, $\sum_{n=1}^{\infty}b_n$ is a series we CHOOSE to compare our given series to. With this test, you always have to choose $\sum_{n=1}^{\infty}b_n$ on your own. Typically, we choose p-series since we know everything about them.)

(8) <u>Argument by Partial Sums</u>:

If $S_p=\sum_{n=1}^{p}a_n$ and $\lim_{p\to\infty}S_p=S<\infty$, then $\sum_{n=1}^{\infty}a_n$ converges to S. Otherwise, if $\lim_{p\to\infty}S_p$ either DNE or goes to some sort of infinity, then $\sum_{n=1}^{\infty}a_n$ diverges.

Note: This is the order of tests I prefer. The other tests will be filled in later.

Note:

The infinite sum operator $\sum_{n=1}^{\infty}$ is a linear operator only on convergent positive series!!!!!

That is,

(a) $\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n$ and

(b) $\sum_{n=1}^{\infty}c\,a_n=c\sum_{n=1}^{\infty}a_n$ (where c is a constant) IF both $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are convergent positive series.

In other words, we can distribute the infinite summation ONLY when we already know the series are each convergent.

<u>AST (Alternating Series Test)</u>:

If we have an alternating series, $\sum_{n=1}^{\infty}(-1)^n a_n$, where $a_n>0$ for all n and if $\{a_n\}$ is decreasing and $\lim_{n\to\infty}a_n=0$, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges (at least conditionally).

Another Note: If you have an alternating series and you do the AST first and find conditional convergence, you STILL have to test for absolute convergence before you can make your final conclusion.

If $f(x)=\sum_{n=0}^{\infty}a_nx^n$ converges on the interval I, then $f'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}$ and $\int_0^x f(t)dt=\sum_{n=0}^{\infty}\frac{a_nx^{n+1}}{n+1}$ also converge on the interior of I.

<u>Taylor's Formula with Remainder</u>:

Assume $f(x)$ is a function with at least (n+1) derivatives existing for each x in an open interval, I, containing a. Then, for each x in that interval, I,

$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x)$

where the remainder (or error) $R_n(x)$ is given by $R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c\in[\min(a,x),\max(a,x)]$.

<u>Power Series to have on your note card</u>:

(along with their convergence sets)

(1) $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \quad \forall x\in(-1,1)$

(2) $\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n} \quad \forall x\in(-1,1]$

(3) $\arctan x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{2n-1} \quad \forall x\in[-1,1]$

(4) $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \forall x\in\mathbb{R}$

<u>Power (Maclaurin) Series to have on your note card</u>:

(5) $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!} \quad \forall x\in\mathbb{R}$

(6) $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(2n)!} \quad \forall x\in\mathbb{R}$

(7) $\sinh x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} \quad \forall x\in\mathbb{R}$

(8) $\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \quad \forall x\in\mathbb{R}$

<u>Taylor's Theorem:</u>

Assume $f(x)$ is a function with derivatives of all orders in some interval $(a-R,a+R)$. The Taylor Series for $f(x)$ is given by

$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots$

on $(a-R,a+R)$ where R is the radius of convergence,

iff

$\lim_{n\to\infty}R_n(x)=0$, i.e. the remainder goes to zero, where the remainder is given by

$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c\in(a-R,a+R)$.

Remember that a Maclaurin series is just a Taylor Series with $a=0$, i.e. the center value is 0.

<u>Taylor's Formula with Remainder</u>:

Assume $f(x)$ is a function with at least (n+1) derivatives existing for each x in an open interval, I, containing a. Then, for each x in that interval, I,

$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x)$

where the remainder (or error) $R_n(x)$ is given by $R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c\in[\min(a,x),\max(a,x)]$.

Chapter 10

Formulas

Cartesian/Rectangular to Polar Coordinates:

$r^2=x^2+y^2$

$\tan\theta=\frac{y}{x}$

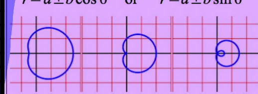
Polar to Cartesian/Rectangular Coordinates:

$x=r\cos\theta$

$y=r\sin\theta$

<u>Limacon</u>:

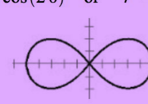
$r=a\pm b\cos\theta$ or $r=a\pm b\sin\theta$



a > b a = b a < b

<u>Lemniscate</u>:

$r^2=\pm a\cos(2\theta)$ or $r^2=\pm a\sin(2\theta)$

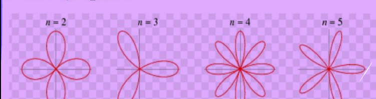


<u>Rose</u>:

$r=a\cos(n\theta)$ or $r=a\sin(n\theta)$

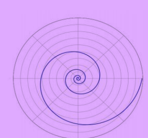
if n is odd, then there are n leaves (or petals)

if n is even, then there are 2n leaves (or petals



<u>Spiral</u>:

$r=a\theta$



<u>Tangent Line Slope:</u>

Given $r=f(\theta)$ curve, slope is

$m=\frac{dy/d\theta}{dx/d\theta}=\frac{f(\theta)\cos\theta+f'(\theta)\sin\theta}{-f(\theta)\sin\theta+f'(\theta)\cos\theta}$

<u>Area "between" two polar curves</u>:

(1) $A=\frac{1}{2}\int_\alpha^\beta \text{radius}^2\,d\theta$

where $\text{radius}^2=(f(\theta))^2$

if it's just one region with only one curve boundary

or possibly

(2) $A=\frac{1}{2}\int_\alpha^\beta(\text{outer radius}^2-\text{inner radius}^2)\,d\theta$

where $\text{outer radius}^2=(f(\theta))^2$ and $\text{inner radius}^2=(g(\theta))^2$