Math5700 Notes
Section 3.1
Functions

Starters:

For problems 1-5, solve for x.

1. $|x+1| = 2x - 4$

2. $|2x+3| \leq 7$

3. $\frac{4x}{2x+3} > 2$

4. $x^3 - 3x^2 - 18x \leq 0$

5. $x = 5 - \sqrt{31 - 9x}$

6. Find solutions for $(x^2 + y^2)^2 - 13(x^2 + y^2) + 36 = 0$ and describe the solution set graphically.

7. Write one equation that describes all points, (x,y), that are four units from (-3, 9) AND six units from (2, 5).

Another few starters:

How many ounces of a 70% oil solution needs to be mixed with 10 ounces of a 40% oil solution in order to obtain a solution that is 50% oil?

Explain the difference between expression, equation, identity, function.

Expression	Equation
Function	Identity

What is a unary operator? Binary operator? Examples?

How do you explain that 0^0 is undefined?

Provisional Definition 1: Let A and B be nonempty sets. We define a *function from A to B*, written $f: A \rightarrow B$, to be a rule that assigns, to each element $a \in A$, an element $f(a) \in B$. The set A is called the *domain of the function* and the set B is called the *codomain of the function*.

Note: $f(a) = b$ denotes that output assigned to *a* depends on input.

Your Turn 1: In high school mathematics, the emphasis is usually on the rule and not on the domain and codomain. For example, we might write down the rule $f(x) = 2^x$ or $g(x) = \frac{1}{x}$ without calling attention to the domain and codomain.

What are reasonable domains and codomains for these functions?
Is there more than one possible answer?
Is there a smallest or largest possible answer?

Your Turn 2: Consider the curve that is the solution set for $x^2 + y^4 = 1$. Imagine we are moving along the curve and as we move, our x and y values change. However, these coordinates are not free to change independently. The fact that we must stay on the curve constrains us, due to the fact that x and y satisfy a relation with respect to each other.

(a) How is the curve a function of x, f(x)? Identify domain, codomain, and rule for function.
(b) How is the curve a function of y, g(y)? Identify domain, codomain, and rule for function.
(c) Draw a piece of this curve that is a function of x.
(d) Draw a piece of this curve that is a function of y.
(e) Draw a piece of this curve that is a function of x and a function of y.

The notion of graph makes sense for functions, and helps with the rigorous definition of function.

(Question: Why does a function of one variable graph into a curve in 2-d? And, if we have a function of two variables, what dimension does it graph into?)

Provisional Definition 2: Let $f: A \rightarrow B$ be a function. We define the *graph of f* to be $\{(a, f(a)) : a \in A\}$. The graph of f is a subset of the Cartesian product A x B.

Definition 3: Let A and B be nonempty sets.

(a) A *relation between A and B* is any subset of A x B.

(b) Let S be a relation from A to B. We say that S is a *function from A to B* if given any $a \in A$, t here is exactly one element of S whose first entry is *a*.

Definition 4: Let $f \subseteq A \, x \, B$ be a function. By definition, *the graph of f* is a synonym for f.

Warning Note: Please be careful with your language! Asking students to graph the function x^3+1 is not clear that you mean x is the input and we have an output variable. It is correct to ask to graph the function f defined by $f(x) = x^3+1$ which explicitly states that x is the input variable and f(x) is the output and thus requires two dimensions to graph the curve/function.

Example 1:

Consider the function $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ given by

$$f(x) = \begin{pmatrix} e^{\frac{-1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{pmatrix}$$

(a) Why is it reasonable to define f(0)=0?

(b) Construct a graph (without a calculator) of this function.

(c) What makes it hard to construct a meaningful graph?

Example 2: Let $f(x)=x^3-3x^2=x^2(x-3)$. Graph this function.
Note the points A(-1, -4), B(0, 0), C(1, -2), D(2, -4), E(3, 0) and a generic point P on the curve. (We can let P be the point (c, f(c)).)

(a) Let g(x) be a function whose graph is obtained from the graph of f by a horizontal shift to the right by 2 units.
Let A', B', C', D', E' be the shifted points (corresponding to A, B, C, D, E respectively) on the graph of g. Where are these new points?

(b) Where is the point P'? Start with P'(c+2, f(c+2)).

(c) Give the function g(x).

Example 3: Follow the ideas of example 2, with the same base function $f(x)=x^3-3x^2=x^2(x-3)$ but perform the following different transformations.

(a) Shift down by 4 units.
(b) Vertical stretch by a factor of 2.
(c) Horizontal shrink by factor of 1/3.
(d) Horizontal reflection (across the vertical axis).
(e) Vertical reflection (across the horizontal axis).

Definition 5: Let $f: A \rightarrow B$ be a function. If S is a subset of A, we define $f[S]=\{f(s): s \in S\}$. The set f[S] is a subset of the codomain, and is called the *image of S under f.*

The set f[A] is called the *range of the function f.* It is a subset of the codomain. The range is just a special case of the image of a subset.

Definition 6: If T is a subset of B, we define $f^{-1}[T]=\{a \in A: f(a) \in T\}$. The set $f^{-1}[T]$ is a subset of A and is called the *inverse image of T under f* or the *pre-image of T under f.*

Example 4: Let $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$, $f(x)=x^2$.

(a) Let T = {4, 9, 0, 3, -2}. Compute $f^{-1}[T]$.

(b) Let T be the interval [2, 60]. Is $-3 \in f^{-1}[T]$?

Example 5: Let $f: A \rightarrow B$ be a function.

(a) Prove that if $S \subseteq A$, then $f^{-1}[f[S]] \supseteq S$.

(b) Prove that if $T \subseteq B$, then $f[f^{-1}[T]] \subseteq T$

Definition 7: Let A be any set. We define the function $I_A : A \to A$ by the rule $I_A(a) = a$ for all a in A. The function I_A is called the *identity function* on A.

Example 6: Let $f : A \to B$ be a function.
(a) What are the domain and codomain of the function $I_B \circ f$?
(b) How can you simplify the rule for $I_B \circ f$?
(c) What similar construction can you make involving I_A ?
(d) Where else in mathematics do we use the word "identity?" How are they all related?

Definition 8: If $f: A \to B$ and $g: B \to A$ are functions, then *f and g are inverses* of each other if $g \circ f = I_A$ and $f \circ g = I_B$, or in other words, if *g(f(a)) = a* for all *a* in *A* and *f(g(b)) = b* for all *b* in *B*.

Proposition:
Let $f: A \to B$ be a function. If g_1 and g_2 are both inverses of *f*, then $g_1 = g_2$.

(This means the inverse function is unique, if it exists.)

Proof? (Hint: Check out $g_1 \circ f \circ g_2$.)

Example 7: A student claims that the inverse of $f(x) = x^3$ is $f^{-1}(x) = \frac{1}{x^3}$. What is the source of their confusion? How do you help this student fix the misunderstanding?

Example 8: Does the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ have an inverse? Justify your answer.

Example 9: Describe how you would explain to a student the process for finding the inverse of the function $f(x) = 2x^5 - 1$.