M2200 Mid 1 Review

- Conditional wording for $p \to q$ (pg 6)
- DeMorgan's Laws (table pg 22)
- logical equivalence tables (pg 24-25)
- Rules of Inference (pg 66)
- Set Identities (pg 124)

(1) $(\bar{q} \wedge (p \to q)) \to \bar{p}$

$\bar{q} \wedge (p \to q) \to \bar{p} \equiv \bar{q} \wedge (\bar{p} \vee q) \to \bar{p}$

$\equiv (\bar{q} \wedge \bar{p}) \vee (\bar{q} \wedge q) \to \bar{p}$

$\equiv (\bar{q} \wedge \bar{p}) \vee F \to \bar{p}$

$\equiv \bar{q} \wedge \bar{p} \to \bar{p}$

If $p \equiv T$, $\bar{p} \equiv F \Rightarrow$

$(\bar{q} \wedge \bar{p}) \equiv F \Rightarrow (\bar{q} \wedge \bar{p}) \equiv F$

and $F \to \bar{p} \equiv T$

If $p \equiv F$, $\bar{p} \equiv T \Rightarrow$

(i) $\bar{q} \equiv T$, then $T \to T \equiv T$

(ii) $\bar{q} \equiv F$, $F \to T \equiv T$

(2) let b = bldg c = ~~campus~~ college r = room domain: colleges in U.S.

$P(r, b, c)$ = r is painted white on c in bldg b.

$\exists b \exists c \forall r \; P(r, b, c)$

negation: $\forall b \forall c \exists r \; \neg P(r, b, c)$

(3) p = healthy q = taste good a = almond c = candy

R = Carol eats it

$\forall x (p(x) \to \neg q(x))$

$p(a)$

$\therefore \neg q(a)$

$R \to q$

$\neg R(a)$

$\neg p(c)$

no conclusion

$p \to \neg q$

$\neg q \to r$

$\therefore p \to r$

$R \to q$

$\neg R(a)$

$\neg q(a)$

$\therefore \neg R(a)$ already

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(4) 3 suspects C = Cooper (one killed)
S = smith
J = jones
W = williams
each say "I didn't kill C." ①

② S: "C was friend of J"
③ " W disliked C."
④ J: "I did not know C."
⑤ "I was out of town day C killed."
⑥ W: "I saw S & J w/ C day of murdr."
⑦ "Either S or J must have killed him."

(a) 2 innocent men telling truth
1 guilty man telling truth or lying.

case 1 Assume S & J innocent. They contradict each other. ② + ④ ⇒ No

case 2 Assume J & W innocent. ⑤ + ⑥ contradictn

case 3 Assume S & W innocent ⇒ J killed C

(b) innocent men do not lie

like part (a) except ∃ also the possibility that all men are innocent, & telling the truth. but there are same contradictns we had for (a)

Also possible more than one is guilty.

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(5) $\{1, -1, \sqrt{5}, -\sqrt{5}\}$

(6) Show $(B-A) \cup (C-A) = (B \cup C) - A$ (can do w/ membership table)

$B - A = B \cap \bar{A}$ $C - A = C \cap \bar{A}$ $(B \cup C) - A = (B \cup C) \cap \bar{A}$

$$(B-A) \cup (C-A) = (B \cap \bar{A}) \cup (C \cap \bar{A})$$
$$= (\bar{A} \cap B) \cup (\bar{A} \cap C)$$
$$= \bar{A} \cap (B \cup C)$$
$$= (B \cup C) \cap \bar{A}$$
$$= (B \cup C) - A$$

(7) (a) If $A - B = A$, then $A \cap B = \phi$.

(b) If $A - B = B - A$, then $A = B$. and $A - B = \phi$.

(8) $A_i = \{\ldots, -2, -1, 0, 1, \ldots i\}$ $A_1 = \{\ldots, -2, -1, 0, 1\}$ $A_2 = \{\ldots, -2, -1, 0, 1, 2\}$

$\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$ $\bigcap_{i=1}^{\infty} A_i = \{\ldots, -2, -1, 0, 1\} = A_1$

(9) (a) $f(x) = 2x + 10$ $f: \mathbb{R} \to \mathbb{R}$

1-1 $f(x) = f(y)$
$2x + 10 = 2y + 10$
$x = y \Rightarrow f$ 1-1

onto $2x + 10 = r$ for some $r \in \mathbb{R}$
$x = \frac{r - 10}{2}$

$\Rightarrow f$ bijective
$f^{-1}(x) = \frac{x - 10}{2}$

(b) $f(x) = x^4 + 1$ not onto: $r = -3$ $x^4 + 1 = -3$ has NS
not 1-1: $f(1) = f(-1)$.

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(9) (cont) (c) $f(x) = \dfrac{x^3+1}{x^3-2}$

1-1 $f(x)=f(y)$ $\quad \dfrac{x^3+1}{x^3-2} = \dfrac{y^3+1}{y^3-2}$

$(x^3+1)(y^3-2) = (y^3+1)(x^3-2)$

$\cancel{x^3y^3} - 2x^3 + y^3 - \cancel{2} = \cancel{x^3y^3} - 2y^3 + x^3 \cancel{-2}$

$3y^3 = 3x^3$

$x=y$ ✓ $\quad$ f is 1-1

onto $\quad r \in \mathbb{R}, \quad r = \dfrac{x^3+1}{x^3-2}$

$r(x^3-2) = x^3+1$

$x^3(r-1) = 1+2r$

$x = \sqrt[3]{\dfrac{1+2r}{r-1}}$ $\quad \Rightarrow$ f onto

$f^{-1}(x) = \sqrt[3]{\dfrac{1+2x}{x-1}}$

(10) Prove $\forall x \in \mathbb{R}, x \geq 0 \quad \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$

pf ① if $x \in \mathbb{Z}^+$, then $\lfloor x \rfloor = x \Rightarrow \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$ ✓

② if $x \notin \mathbb{Z}^+$, then we can write x as

$x = n^2 + m + \varepsilon$ $\quad$ n^2 is largest perfect square $< x$

$m \in \mathbb{Z}^+, \quad \varepsilon \in [0,1)$

$\Rightarrow \lfloor x \rfloor = \lfloor n^2+m+\varepsilon \rfloor$

$= n^2+m$

$\sqrt{\lfloor x \rfloor} = \sqrt{n^2+m}$ and $\sqrt{x} = \sqrt{n^2+m+\varepsilon}$

$\Rightarrow n < \sqrt{\lfloor x \rfloor} < n+1$ and $n < \sqrt{x} < n+1$

$\Rightarrow \lfloor \sqrt{\lfloor x \rfloor} \rfloor = n$ and $\lfloor \sqrt{x} \rfloor = n$ //

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(11) Prove: n odd $\Rightarrow n^2$ odd.

Pf If n odd, then $\exists\ k \in \mathbb{Z} \ni n = 2k+1$ by Defn.

$$(2k+1)^2 = 4k^2 + 4k + 1 = 2\underbrace{(2k^2+2k)}_{\in \mathbb{Z}} + 1 \Rightarrow n^2 \text{ odd} //$$

(13) Claim n^2 divisible by 3 $\Rightarrow$ n divisible by 3. (assume $n \in \mathbb{Z}$).

Pf Assume n is not divisible by 3.

Then $\exists\ m \in \mathbb{Z},\ r \in \{1,2\} \ni n = 3m+r$

$\Rightarrow n^2 = (3m+r)^2 = 9m^2 + 6mr + r^2$ and $r^2 = 1$ or 4 both of which are not divisible by 3.

$= 3(3m^2 + 2mr) + r^2$

$\Rightarrow n^2$ not divisible by 3 $\Rightarrow$ contradiction $\Rightarrow n$ is divisible by 3 //

(14) Prove $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

Pf Let $x = m + \varepsilon$ $m \in \mathbb{Z}$, $\varepsilon \in [0,1)$

case ① $\varepsilon < \frac{1}{2}$

$\lfloor 2x \rfloor = \lfloor 2m + 2\varepsilon \rfloor = 2m$

$\lfloor x \rfloor = \lfloor m + \varepsilon \rfloor = m$

$\lfloor x + \frac{1}{2} \rfloor = \lfloor m + \varepsilon + \frac{1}{2} \rfloor = m$

$\Rightarrow \lfloor 2x \rfloor = 2m = m + m = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$ //

case ② $\varepsilon \geq 1/2$

$\lfloor 2x \rfloor = \lfloor 2m + 2\varepsilon \rfloor = 2m + 1$

$\lfloor x \rfloor = m$

$\lfloor x + \frac{1}{2} \rfloor = \lfloor m + \varepsilon + \frac{1}{2} \rfloor = m + 1$

$\lfloor 2x \rfloor = 2m + 1 = m + m + 1$
$= \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$ //

(15) Prove $n^2 \geq 2n$ $\forall n \in \mathbb{N},\ n \geq 2$

Pf $n \geq 2 \iff n - 2 \geq 0$

$n(n-2) \geq 0$

$n^2 \geq 2n$ //

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(16) Between any two distinct $\mathbb{Q}$ #s, $\exists$ a ratnl #.

pf let x and $y \in \mathbb{Q}$. $\Rightarrow x = \frac{p}{q}$ $y = \frac{a}{b}$, $a, b, p, q \in \mathbb{Z}$

$x \neq y$. $b \neq 0$, $q \neq 0$.

Then $\frac{x+y}{2} = \frac{\frac{p}{q} + \frac{a}{b}}{2} = \frac{pb + aq}{2bq} \in \mathbb{Q}$ and $bq \neq 0$

$\Rightarrow \frac{x+y}{2} \in \mathbb{Q}$ and

$x < \frac{x+y}{2} < y$.