Math 2200 HW #9 Graded Problems

$\boxed{4.2}$ # 2, 7, 12, 26, 30

$\boxed{4.3}$ # 4, 6, 8, 12, 13

4.2

(2) Show (w/ strong induction) all dominoes fall in an ~~so~~ arrangement of dominoes if you know that the first 3 fall, and when one domino falls, the domino 3 farther down falls.

Pf (1) We already were given that it's true for $n=1,2,3$.

(2) Assume it's true for $n=1,2,3,\dots,m$, i.e. that the first m dominoes all fall.
Then for $m+1$ case, we know the $(m+1)-3 = m-2$ domino fell $\Rightarrow m+1$ domino also fell. //

(7) Which amts of money can be formed using \$2 bills & \$5 bills?

Claim $2a+5b=m$ for some $a,b \in W$, then m can be any integer ≥ 4.

Pf (1) check $m=4$ case + $m=5$ case.
if $m=4$, $a=2, b=0$ works
if $m=5$, $a=0, b=1$ works

(2) Assume statement is true $\forall$ $m=4,5,\dots,n$. Then we can find $a_m, b_m \in W$ $\ni$ $2a_m+5b_m=m$.
check $n+1$ case. We know $n+1 = 2a_n+5b_n+1$ from induction assumption.

also $n+1=(n-1)+2 = (2a_{n-1}+5b_{n-1})+2$
$= 2(a_{n-1}+1)+5b_{n-1}$ $\Rightarrow$ $a_{n+1}=a_{n-1}+1$
$b_{n+1}=b_{n-1}$ //

4.2 (#12) Show every $n \in \mathbb{Z}^+$ can be written as sum of distinct powers of 2, i.e. as sum of $2^0=1, 2^1=2, \ldots$

Claim $n \in \mathbb{Z}^+$

$n = a_k 2^k + a_{k-1} 2^{k-1} + \cdots + a_1 2 + a_0$ for some $k \in \mathbb{Z}^+$, where $a_i = 0$ or 1 $\forall\, i = 0, 1, \ldots, k$

pf ① $n=1$ case $1 = 2^0$ ✓

② Assume true $\forall\, n = 1, 2, \ldots, m$. Check $m+1$ case.

$m = a_p 2^p + a_{p-1} 2^{p-1} + \cdots + a_2 2^2 + a_1 2 + a_0$ where $a_i = 0$ or 1, $i = 0, \ldots, p$
(induction assumption)

$\Rightarrow$ $m+1 = a_p 2^p + a_{p-1} 2^{p-1} + \cdots + a_2 2^2 + a_1 2 + a_0 + 1$

case 1: if $a_0 = 0$ (m even), then $a_0 + 1 = 1 = 1(2^0)$

$\Rightarrow m+1 = a_p 2^p + \cdots + a_2 2^2 + a_1 2 + 1$ ✓

case 2: if $a_0 = 1$ (m odd), then $m+1$ even.

$\Rightarrow \frac{m+1}{2} = b \in \mathbb{Z}^+$, $b < m$ $\Rightarrow$ b can be written as sum of powers of 2, i.e.

$b = a_{rb} 2^r + a_{r-1,b} 2^{r-1} + \cdots + a_{1b} 2 + a_{0b}$

and $m+1 = 2b = a_{rb} 2^{r+1} + a_{r-1,b} 2^r + \cdots + a_{1b} 2^2 + a_{0b} 2$ //

(26) $P(n)$ is propositnl fn. Determine for which nonnegative integers n $P(n)$ must be true if

(a) $P(0)$ is true; $\forall\, n \in W$, $P(n) \to P(n+2)$ true.
then $P(n)$ is true for all even values of n,
$P(0), P(2), P(4), \ldots$

(b) $P(0)$ true; $\forall\, n \in W$ $P(n) \to P(n+3)$ true.
$P(0), P(3), P(6), P(9), \ldots$ i.e. true $\forall$ values of n that are multiple of 3.

4.2 (26) cont

(c) $P(0)$ & $P(1)$ true; $\forall n \in \mathbb{W}$, if $P(n)$ & $P(n+1)$ are true, then $P(n+2)$ true.
get all $P(0), P(1), P(2), P(3), P(4), \ldots$ true $\forall n \in \mathbb{W}$.

(d) $P(0)$ true; $\forall n \in \mathbb{W}$ $P(n)$ true $\Rightarrow$ $P(n+2)$ & $P(n+3)$ true.
then we get $P(0), P(2), P(3), P(4), P(5), \ldots$
i.e. everything except $P(1)$ is true.

(30) Find flaw.

claim $a^n = 1$ $\forall n \in \mathbb{W}$, $a \in \mathbb{R}$, $a \neq 0$.

Pf (1) $a^0 = 1$ true by defn a^0.

(2) Assume $a^j = 1$ $\forall$ $j \leq k$, $j \in \mathbb{W}$.

$$\Rightarrow a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1 \quad //$$

* we assumed $k \geq 1$ for this step which means we should have also checked $a^1 = 1$ (which is not true, of course).

4.3 (#4) Find $f(2), f(3), f(4), f(5)$ for
$f(0) = f(1) = 1$ & $\forall n = 1, 2, \ldots$

(a) $f(n+1) = f(n) - f(n-1)$

$f(2) = f(1) - f(0) = 0$

$f(3) = f(2) - f(1) = -1$

$f(4) = f(3) - f(2) = -1 - 0 = -1$

$f(5) = f(4) - f(3) = -1 - (-1) = 0$

#4 (cont)

(b) $f(n+1) = f(n)\, f(n-1)$

$f(2) = f(1) f(0) = 1$

$f(3) = f(2) f(1) = 1$

$f(4) = f(3) f(2) = 1$

$f(5) = f(4)(f(3)) = 1$

(c) $f(n+1) = [f(n)]^2 + [f(n-1)]^3$

$f(2) = [f(1)]^2 + [f(0)]^3 = 2$

$f(3) = [f(2)]^2 + [f(1)]^3 = 4 + 1 = 5$

$f(4) = [f(3)]^2 + [f(2)]^3 = 25 + 8 = 33$

$f(5) = [f(4)]^2 + [f(3)]^3 = 33^2 + 125 = 1214$

(d) $f(n+1) = \dfrac{f(n)}{f(n-1)}$

$f(2) = \dfrac{f(1)}{f(0)} = 1$

$f(3) = \dfrac{f(2)}{f(1)} = 1$

$f(4) = \dfrac{f(3)}{f(2)} = 1$

$f(5) = \dfrac{f(4)}{f(3)} = 1$

#6 valid recursive fn from $\mathbb{Z}^+$ to $\mathbb{Z}$? if f well-defined, find formula for $f(n)$ & prove it.

(a) $f(0) = 1$, $f(n) = -f(n-1)$ $\forall\, n \geq 1$ valid

n	f
0	1
1	-1
2	1
3	-1

$\Rightarrow$ claim $f(n) = (-1)^n$

#6 cont

(a) (cont)

Pf ① $f(0)=1$ and $(-1)^0=1$ ✓

② Assume true ~~for~~ n, i.e. $f(n)=(-1)^n$.

check $n+1$ case.

we know $f(n+1) = -f(n)$ by defn

$= -(-1)^n$ by inductn assumptn

$= (-1)^{n+1}$ //

(b) $f(0)=1$, $f(1)=0$, $f(2)=2$, $f(n)=2f(n-3)$ $\forall n \geq 3$. valid

n	f	
0	1	$=2^0$
1	0	
2	2	$=2^1$
3	2	$=2^1$
4	0	
5	4	$=2^2$
6	4	$=2^2$
7	0	
8	8	$=2^3$
9	8	$=2^3$
10	0	
11	16	$=2^4$

claim $f(n) = \begin{cases} 2^{n/3} & n \equiv 0 \bmod 3 \\ 0 & n \equiv 1 \bmod 3 \\ 2^{\frac{n+1}{3}} & n \equiv 2 \bmod 3 \end{cases}$

Pf ① $n=0$, $f(0)=1$ $2^{0/3}=1$ ✓

$n=1$, $f(1)=0$, 0 ✓

$n=2$; $f(2)=2$, $2^{\frac{2+1}{3}}=2^1=2$ ✓

② Assume claim is true $\forall n = 0,1,2,\ldots,m$.

check $m+1$ case. $f(m+1) = 2f(m+1-3) = 2f(m-2)$ by defn

(i) if $m+1 \equiv 0 \bmod 3$, then $m-2 \equiv 0 \bmod 3$ $\Rightarrow f(m+1) = 2(2^{\frac{m-2}{3}})$

$= 2^{\frac{3+m-2}{3}} = 2^{\frac{m+1}{3}}$ //

(ii) if $m+1 \equiv 1 \bmod 3$, $f(m+1) = 2f(m-2)$

then $m-2 \equiv 1 \bmod 3$ $\Rightarrow f(m+1) = 2(0) = 0$

(iii) if $m+1 \equiv 2 \bmod 3$, $f(m+1) = 2f(m-2)$

then $m-2 \equiv 2 \bmod 3$ $\Rightarrow f(m+1) = 2(2^{\frac{m-2+1}{3}}) = 2^{\frac{m+2}{3}}$ //

4.3
#6 cont

(c) $f(0)=0,\ f(1)=1,\ f(n)=2(n+1) \quad \forall n\geq 2$

unvalid because it needs the next term in sequence before it is defined

(d) $f(0)=0,\ f(1)=1,\ f(n)=2f(n-1) \quad \forall n\geq 1$

$f(1)=1$ but according to recursive formula
$f(1)=2f(0)=2(0)=0$

$\Rightarrow$ unvalid

(e) $f(0)=2,\ f(n)=\begin{cases} f(n-1) & n \text{ odd}, \ n\geq 1 \\ 2f(n-2) & n \text{ even}, \ n\geq 2 \end{cases}$ valid

n	f
0	$2=2^1$
1	$2=2^1$
2	$4=2^2$
3	$4=2^2$
4	$8=2^3$
5	$8=2^3$
6	$16=2^4$

claim $f(n)=2^{\lceil \frac{n+1}{2} \rceil}$

pf ① $f(0)=2$ given
$2^{\lceil \frac{0+1}{2} \rceil}=2^1=2$

(if want to check
$f(1)=2$
$2^{\lceil \frac{1+1}{2} \rceil}=2^1=2$ ✓)

② Assume true for $m=1,2,\dots,n$

check $n+1$ case.

By defn, $f(n+1)=\begin{cases} f(n), & n+1 \text{ odd} \\ 2f(n-1), & n+1 \text{ even} \end{cases}$

$=\begin{cases} 2^{\lceil \frac{n+1}{2} \rceil} & n+1 \text{ odd} \\ 2\left(2^{\lceil \frac{n-1+1}{2} \rceil}\right) & n+1 \text{ even} \end{cases}$ by Induction Assumption

$=\begin{cases} 2^{\lceil \frac{n+1}{2} \rceil} & n+1 \text{ odd} \\ 2^{\lceil \frac{n}{2} \rceil+1} & n+1 \text{ even} \end{cases} = \begin{cases} 2^{\lceil \frac{n+1}{2} \rceil} & n+1 \text{ odd} \\ 2^{\lceil \frac{n+1}{2} \rceil} & n+1 \text{ even} \end{cases}$

$=2^{\lceil \frac{n+1}{2} \rceil}$

//

4.3 #8 Give recursive defn of $\{a_n\}$, $n=1,2,3,\ldots$ if

★ there will be lots of right answers here

(a) $a_n = 4n-2$ $\qquad a_1 = 4(1)-2 = 2$

So $a_1 = 2$, $\quad a_n = a_{n-1} + 4$

(b) $a_n = 1+(-1)^n$ $\qquad a_1 = 1+-1 = 0 \qquad a_2 = 1+(-1)^2 = 2$

$\Rightarrow a_1 = 0,\ a_2 = 2,\ a_n = a_{n-2} \quad \forall n \geq 3$

(c) $a_n = n(n+1)$ $\qquad a_1 = 1(2) = 2$

$\Rightarrow a_1 = 2,\quad a_n = a_{n-1} + 2n$ $\qquad$ (since I know $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$)

(d) $a_n = n^2$

n	a_n
1	1
2	4
3	9
4	16
5	25

$+3,\ +5,\ +7,\ +9$

$a_1 = 1$
$a_n = a_{n-1} + 2n - 1$

#12 Claim $\quad f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1} \quad n \in \mathbb{Z}^+$.

$\iff \sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$

Defn $\quad f_0 = 0,\ f_1 = 1,$
$f_n = f_{n-1} + f_{n-2}$
$n = 2, 3, 4, \ldots$

Pf ① $n=1$ case. $\qquad \sum_{i=1}^{1} f_i^2 = f_1^2 = 1^2 = 1$

and $f_1 f_2 = 1(1+0) = 1$ ✓

② Assume true $\forall\ m = 1, 2, \ldots, n$, i.e. $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$.

Check $n+1$ case. $\qquad \sum_{i=1}^{n} f_i^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2$

$= f_{n+1}(f_n + f_{n+1})$

$= f_{n+1}(f_{n+2})$ $\qquad$ by defn

$= f_{n+1} f_{n+2}$ //

4.3 #13 claim $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ $n \in \mathbb{Z}^+$.

$$\iff \sum_{i=1}^{n} f_{2i-1} = f_{2n}$$

pf ① $n=1$ case. $\sum_{i=1}^{1} f_{2i-1} = f_1 = 1$

and $f_{2n} = f_2 = 1$ ✓

② Assume true $\forall$ $n = 1, 2, \ldots, n$, i.e. $\sum_{i=1}^{n} f_{2i-1} = f_{2n}$.

check $n+1$ case.

$$\sum_{i=1}^{n} f_{2i-1} + f_{2(n+1)-1} = f_{2n} + f_{2(n+1)-1}$$

$$\sum_{i=1}^{n+1} f_{2i-1} = f_{2n} + f_{2n+1}$$

$$= f_{2n+2} \text{ (by defn) //}$$