28. *Volume*
    (a) A square piece of cardboard 36 centimeters on a side is to be formed into a rectangular box by cutting squares with length x from each corner and folding up the sides. What is the maximum volume possible for the box?
    (b) Show that if the piece of cardboard is k centimeters on each side, cutting squares of size $k/6$ and folding up the sides gives the maximum volume.

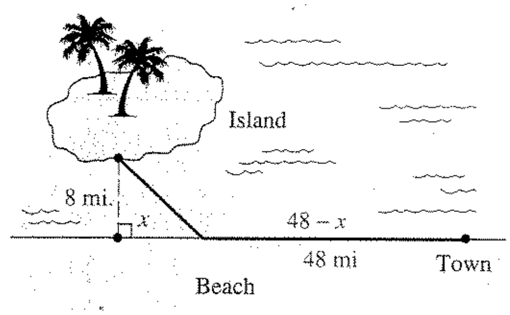
29. *Revenue* The owner of an orange grove must decide when to pick one variety of oranges. She can sell them for $8 a bushel if she sells them now, with each tree yielding an average of 5 bushels. The yield increases by half a bushel per week for the next 5 weeks, but the price per bushel decreases by $0.50 per bushel each week. When should the oranges be picked for maximum return?

30. *Minimum material*
    (a) A box with an open top and a square base is to be constructed to contain 4000 cubic inches. Find the dimensions that will require the minimum amount of material to construct the box.
    (b) A box with an open top and a square base is to be constructed to contain k cubic inches. Show that the minimum amount of material is used to construct the box when each side of the base is

$$x = (2k)^{1/3} \text{ and the height is } y = \left(\frac{k}{4}\right)^{1/3}$$

31. *Minimum cost* A printer has a contract to print 100,000 posters for a political candidate. He can run the posters by using any number of plates from 1 to 30 on his press. If he uses x metal plates, they will produce x copies of the poster with each impression of the press. The metal plates cost $2.00 to prepare, and it costs $12.50 per hour to run the press. If the press can make 1000 impressions per hour, how many metal plates should the printer make to minimize costs?

32. *Shortest time* A vacationer on an island 8 miles offshore from a point that is 48 miles from town must travel to town occasionally. (See the figure.) The vacationer has a boat capable of traveling 30 mph and can go by auto along the coast at 55 mph. At what point should the car be left to minimize the time it takes to get to town?



33. *DVD players* The sales revenue for DVD players is given by

$$R(t) = -0.094t^3 + 0.680t^2 + 0.7175t + 0.826$$

in billions of dollars, where t is the number of years past 1999. Use a graphing utility to find the year when sales revenue is maximized.

10.5 Rational Functions: More Curve Sketching

OBJECTIVES

- To locate horizontal asymptotes
- To locate vertical asymptotes
- To sketch graphs of functions that have vertical and/or horizontal asymptotes

Application Preview

Suppose that the total cost of producing a shipment of a product is

$$C(x) = 5000x + \frac{125{,}000}{x}, \quad x > 0$$

where x is the number of machines used in the production process. To find the number of machines that will minimize the total cost, we find the minimum value of this rational function. (See Example 3.) The graph of this function contains a vertical asymptote at $x = 0$. We will discuss graphs and applications involving asymptotes in this section.

The procedures for using the first-derivative test and the second-derivative test are given in previous sections, but none of the graphs discussed in those sections contains vertical asymptotes or horizontal asymptotes. In this section, we consider how to use information about asymptotes along with the first and second derivatives, and we present a unified approach to curve sketching.

Asymptotes

In Section 2.4, "Special Functions and Their Graphs," we first discussed asymptotes and saw that they are important features of the graphs that have them. Then, in our discussion of limits in Sections 9.1 and 9.2, we discovered the relationship between certain limits and asymptotes. The formal defination of vertical asymptotes uses limits.

Vertical Asymptote

The line $x = x_0$ is a **vertical asymptote** of the graph of $y = f(x)$ if the values of $f(x)$ approach $+\infty$ or $-\infty$ as x approaches x_0 (from the left or the right).

From our work with limits, recall that a vertical asymptote will occur on the graph of a function at an x-value at which the denominator (but not the numerator) of the function is equal to zero. These observations allow us to determine where vertical asymptotes occur.

Vertical Asymptote of a Rational Function

The graph of the rational function

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$ if $g(c) = 0$ and $f(c) \neq 0$.

Because a horizontal asymptote tells us the behavior of the values of the function (y-coordinates) when x increases or decreases without bound, we use limits at infinity to determine the existence of horizontal asymptotes.

Horizontal Asymptote

The graph of a rational function $y = f(x)$ will have a **horizontal asymptote** at $y = b$, for a constant b, if

$$\lim_{x\to+\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Otherwise, the graph has no horizontal asymptote.

For a rational function f, $\lim_{x\to+\infty} f(x) = b$ if and only if $\lim_{x\to-\infty} f(x) = b$, so we only need to find one of these limits to locate a horizontal asymptote. In Problems 37 and 38 in the 9.2 Exercises, the following statements regarding horizontal asymptotes of the graphs of rational functions were proved.

Horizontal Asymptotes of Rational Functions

Consider the rational function $y = \dfrac{f(x)}{g(x)} = \dfrac{a_nx^n + \cdots + a_1x + a_0}{b_mx^m + \cdots + b_1x + b_0}$.

1. If $n < m$ (that is, if the degree of the numerator is less than that of the denominator), a horizontal asymptote occurs at $y = 0$ (the x-axis).
2. If $n = m$ (that is, if the degree of the numerator equals that of the denominator), a horizontal asymptote occurs at $y = \dfrac{a_n}{b_m}$ (the ratio of the leading coefficients).
3. If $n > m$ (that is, if the degree of the numerator is greater than that of the denominator), there is no horizontal asymptote.

EXAMPLE 1 ***Vertical and Horizontal Asymptotes***

Find any vertical and horizontal asymptotes for

(a) $f(x) = \dfrac{2x - 1}{x + 2}$

(b) $f(x) = \dfrac{x^2 + 3}{1 - x}$

Solution

(a) The denominator of this function is 0 at $x = -2$, and because this value does not make the numerator 0, there is a vertical asymptote at $x = -2$. Because the function is rational, with the degree of the numerator equal to that of the denominator and with the ratio of the leading coefficients equal to 2, the graph of the function has a horizontal asymptote at $y = 2$. The graph is shown in Figure 10.42(a).

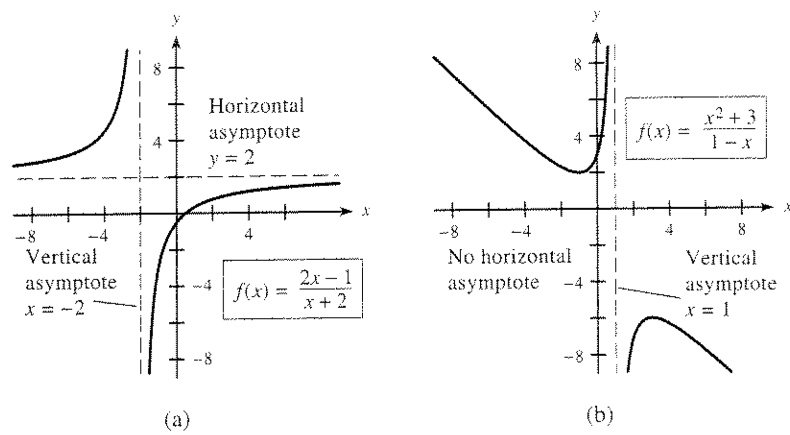


Figure 10.42

(b) At $x = 1$, the denominator of $f(x)$ is 0 and the numerator is not, so a vertical asymptote occurs at $x = 1$. The function is rational with the degree of the numerator greater than that of the denominator, so there is no horizontal asymptote. The graph is shown in Figure 10.42(b).

More Curve Sketching

We now extend our first- and second-derivative techniques of curve sketching to include functions that have asymptotes.

In general, the following steps are helpful when we sketch the graph of a function.

1. Determine the domain of the function. The domain may be restricted by the nature of the problem or by the equation.
2. Look for vertical asymptotes, especially if the function is a rational function.
3. Look for horizontal asymptotes, especially if the function is a rational function.
4. Find the relative maxima and minima by using the first-derivative test or the second-derivative test.
5. Use the second derivative to find the points of inflection if this derivative is easily found.
6. Use other information (intercepts, for example) and plot additional points to complete the sketch of the graph.

EXAMPLE 2 *Graphing with Asymptotes*

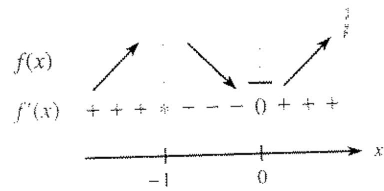
Sketch the graph of the function $f(x) = \dfrac{x^2}{(x+1)^2}$.

Solution

1. The domain is the set of all real numbers except $x = -1$.
2. Because $x = -1$ makes the denominator 0 and does not make the numerator 0, there is a vertical asymptote at $x = -1$.
3. Because $\dfrac{x^2}{(x+1)^2} = \dfrac{x^2}{x^2 + 2x + 1}$
the function is rational with the degree of the numerator equal to that of the denominator and with the ratio of the leading coefficients equal to 1. Hence, the graph of the function has a horizontal asymptote at $y = 1$.
4. To find any maxima and minima, we first find $f'(x)$.

$$f'(x) = \frac{(x+1)^2(2x) - x^2[2(x+1)]}{(x+1)^4}$$
$$= \frac{2x(x+1)[(x+1) - x]}{(x+1)^4}$$
$$= \frac{2x}{(x+1)^3}$$

Thus $f'(x) = 0$ when $x = 0$ (and $y = 0$), and $f'(x)$ is undefined at $x = -1$ (where the vertical asymptote occurs). Testing $f'(x)$ on either side of $x = 0$ and $x = -1$ gives the following sign diagram. The sign diagram for f' shows that the critical point (0, 0) is a relative minimum and shows how the graph approaches the vertical asymptote at $x = -1$.



*$x = -1$ is a vertical asymptote.

5. The second derivative is

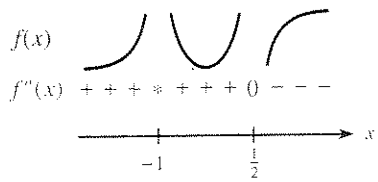
$$f''(x) = \frac{(x+1)^3(2) - 2x[3(x+1)^2]}{(x+1)^6}$$

Factoring $(x+1)^2$ from the numerator and simplifying give

$$f''(x) = \frac{2 - 4x}{(x+1)^4}$$

We can see that $f''(0) = 2 > 0$, so the second-derivative test also shows that (0, 0) is a relative minimum. We see that $f''(x) = 0$ when $x = \frac{1}{2}$. Checking $f''(x)$ between

$x = -1$ (where it is undefined) and $x = \frac{1}{2}$ shows that the graph is concave up on this interval. Note also that $f''(x) < 0$ for $x > \frac{1}{2}$, so the point $(\frac{1}{2}, \frac{1}{9})$ is a point of inflection. Also see the sign diagram for $f''(x)$.



*$x = -1$ is a vertical asymptote.

6. To see how the graph approaches the horizontal asymptote, we check $f(x)$ for large values of $|x|$.

$$f(-100) = \frac{(-100)^2}{(-99)^2} = \frac{10{,}000}{9{,}801} > 1, \quad f(100) = \frac{100^2}{101^2} = \frac{10{,}000}{10{,}201} < 1$$

Thus the graph has the characteristics shown in Figure 10.43(a). The graph is shown in Figure 10.43(b).

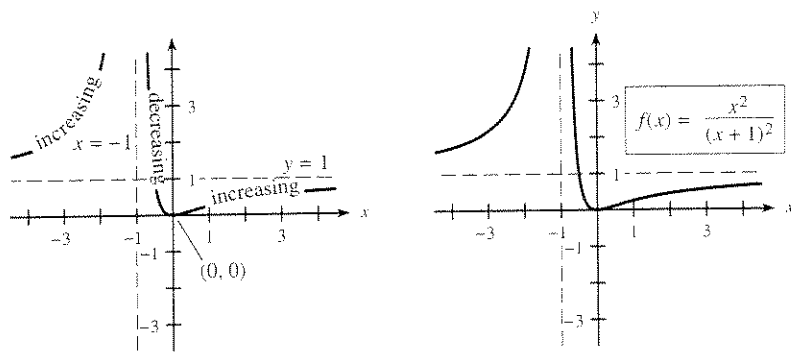


Figure 10.43 (a) (b)

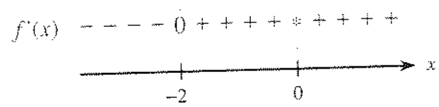
When we wish to learn about a function $f(x)$ or sketch its graph, it is important to understand what information we obtain from $f(x)$, from $f'(x)$, and from $f''(x)$. The following summary may be helpful.

Summary

Source	Information Provided
$f(x)$	y-coordinates; horizontal asymptotes, vertical asymptotes; domain restrictions
$f'(x)$	Increasing $[f'(x) > 0]$; decreasing $[f'(x) < 0]$; critical points $[f'(x) = 0$ *or* $f'(x)$ undefined$]$; sign-diagram tests for maxima and minima
$f''(x)$	Concave up $[f''(x) > 0]$; concave down $[f''(x) < 0]$; possible points of inflection $[f''(x) = 0$ or $f''(x)$ undefined$]$; sign-diagram tests for points of inflection; second-derivative test for maxima and minima

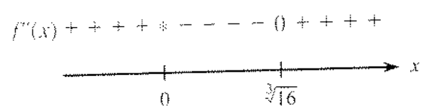
● Checkpoint

1. Let $f(x) = \dfrac{2x + 10}{x - 1}$ and decide whether the following are true or false.
   (a) $f(x)$ has a vertical asymptote at $x = 1$.
   (b) $f(x)$ has $y = 2$ as its horizontal asymptote.
2. Let $f(x) = \dfrac{x^3 - 16}{x} + 1$; then $f'(x) = \dfrac{2x^3 + 16}{x^2}$ and $f''(x) = \dfrac{2x^3 - 32}{x^3}$. Use these to determine whether the following are true or false.
   (a) There are no asymptotes.
   (b) $f'(x) = 0$ when $x = -2$
   (c) A partial sign diagram for $f'(x)$ is



*means $f'(0)$ is undefined.

   (d) There is a relative minimum at $x = -2$.
   (e) A partial sign diagram for $f''(x)$ is



*means $f''(0)$ is undefined.

   (f) There are points of inflection at $x = 0$ and $x = \sqrt[3]{16}$.

EXAMPLE 3 *Production Costs (Application Preview)*

Suppose that the total cost of producing a shipment of a certain product is

$$C(x) = 5000x + \frac{125{,}000}{x}, \qquad x > 0$$

where x is the number of machines used in the production process.

(a) Graph this total cost function, and identify any asymptotes.
(b) How many machines should be used to minimize the total cost?

Solution

(a) Writing this function with all terms over a common denominator gives

$$C(x) = 5000x + \frac{125{,}000}{x} = \frac{5000x^2 + 125{,}000}{x}$$

The domain of $C(x)$ does not include 0, and $C \to +\infty$ as $x \to 0^+$, so there is a vertical asymptote at $x = 0$. Thus the cost increases without bound as the number of machines used in the process approaches zero. The function decreases initially, but eventually increases, and (because the numerator has a higher degree than the denominator) there is no horizontal asymptote (see Figure 10.44).

(b) Finding the derivative of $C(x)$ gives

$$C'(x) = 5000 - \frac{125{,}000}{x^2} = \frac{5000x^2 - 125{,}000}{x^2}$$

Setting $C'(x) = 0$ and solving for x gives the critical values of x.

$$0 = \frac{5000(x+5)(x-5)}{x^2}$$

$$x = 5 \quad \text{or} \quad x = -5$$

Because $C''(x) = 250{,}000x^{-3} = \dfrac{250{,}000}{x^3}$ is positive for all $x > 0$, using 5 machines minimizes the cost at $C(5) = 50{,}000$ (see Figure 10.44).

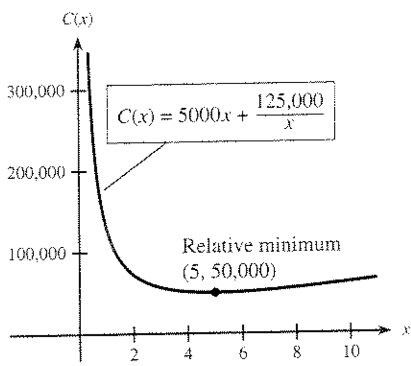


Figure 10.44

Calculator Note If a graphing calculator is not available, the procedures previously outlined in this section are necessary to generate a complete and accurate graph. With a graphing calculator, the graph of a function is easily generated as long as the viewing window dimensions are appropriate. Although a graphing calculator may reveal the existence of asymptotes, it cannot always precisely locate them. Also, we sometimes need information provided by derivatives to obtain a window that shows all features of a graph.

● EXAMPLE 4 *Horizontal and Vertical Asymptotes*

Figure 10.45 shows the graph of $f(x) = \dfrac{71x^2}{28(3 - 2x^2)}$.

(a) Determine whether the function has horizontal or vertical asymptotes, and estimate where they occur.
(b) Check your conclusions to part (a) analytically.
(c) Discuss which method is more accurate.

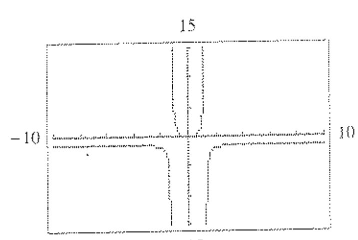


Figure 10.45

Solution

(a) The graph appears to have a horizontal asymptote somewhere between $y = -1$ and $y = -2$, perhaps near $y = -1.5$. Also, there are two vertical asymptotes located approximately at $x = 1.25$ and $x = -1.25$.

(b) The function

$$f(x) = \frac{71x^2}{28(3 - 2x^2)} = \frac{71x^2}{84 - 56x^2}$$

is a rational function with the degree of the numerator equal to that of the denominator and with the ratio of the leading coefficients equal to $-71/56$. Thus the graph of the function has a horizontal asymptote at $y = -71/56$.

Vertical asymptotes occur at x-values where $28(3 - 2x^2) = 0$, or

$$3 - 2x^2 = 0$$
$$3 = 2x^2$$
$$\frac{3}{2} = x^2$$
$$\pm\sqrt{\frac{3}{2}} = x \quad \text{or} \quad x \approx \pm 1.225$$

(c) The analytic method is more accurate, of course, because asymptotes reveal extreme behavior of the function either vertically or horizontally. An accurate graph shows all features, but not necessarily the details of any feature. Despite this, our estimates from the graph were not too bad.

Even with a graphing calculator, sometimes analytic methods are needed to determine an appropriate viewing window.

• EXAMPLE 5 *Graphing with Technology*

The standard viewing window of the graph of $f(x) = \frac{x + 10}{x^2 + 300}$ appears blank (check and see). Find any asymptotes, maxima, and minima, and determine an appropriate viewing window. Sketch the graph.

Solution

Because $x^2 + 300 = 0$ has no real solution, there are no vertical asymptotes. The function is rational with the degree of the numerator less than that of the denominator, so the horizontal asymptote is $y = 0$, which is the x-axis.

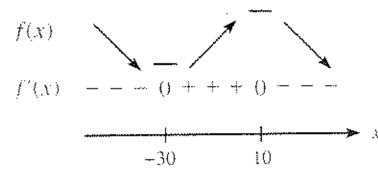
We then find an appropriate viewing window by locating the critical points.

$$f'(x) = \frac{(x^2 + 300)(1) - (x + 10)(2x)}{(x^2 + 300)^2}$$
$$= \frac{x^2 + 300 - 2x^2 - 20x}{(x^2 + 300)^2} = \frac{300 - 20x - x^2}{(x^2 + 300)^2}$$

$f'(x) = 0$ when the numerator is zero. Thus

$$300 - 20x - x^2 = 0$$
$$0 = x^2 + 20x - 300$$
$$0 = (x + 30)(x - 10)$$
$$x + 30 = 0 \qquad x - 10 = 0$$
$$x = -30 \qquad x = 10$$

The critical points are $x = -30$, $y = -\frac{1}{60} \approx -0.01666667$ and $x = 10$, $y = \frac{1}{20} = 0.05$. A sign diagram for $f'(x)$ is shown at the right.



Without using the information above, a graphing utility may not give a useful graph. An x-range that includes -30 and 10 is needed. Because $y = 0$ is a horizontal asymptote, these relative extrema are absolute, and the y-range must be quite small for the shape of the graph to be seen clearly. Figure 10.46 shows the graph.

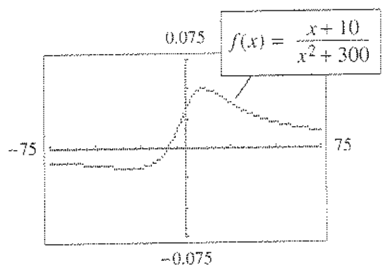


Figure 10.46

● **EXAMPLE 6** ***Profit***

A profit function for a product is given by

$$P(x) = \frac{-x^2 + 16x - 4}{4x^2 + 16}, \qquad \text{for } x \geq 0$$

where x is in thousands of units and $P(x)$ is in billions of dollars. Because of fixed costs, profit is negative when fewer than 255 units are produced and sold. Will a loss occur at any other level of production and sales?

Solution

Looking at the graph of this function over the range $0 \leq x \leq 10$ (see Figure 10.47(a)), it is not clear whether the graph will eventually cross the x-axis. To see whether a loss ever occurs and to see what profit is approached as the number of units produced and sold becomes large, we consider the behavior of $P(x)$ as $x \to \infty$ (i.e., whether $P(x)$ has a horizontal asymptote).

Note that $P(x)$ is a rational function, and from the highest-power terms in the numerator and denominator we see that $y = -\frac{1}{4}$ is a horizontal asymptote. Thus a loss of $\$\frac{1}{4}$ billion is approached as the number of units increases without bound. Figure 10.47(b) shows that the graph does cross the x-axis, where $-x^2 + 16x - 4 = 0$. Using ZERO, SOLVER, the Quadratic Formula, or a program gives $P(x) = 0$ at $x \approx 0.254$ and $x \approx 15.7459$. Thus if 15,746 units or more are produced and sold, the profit is negative.

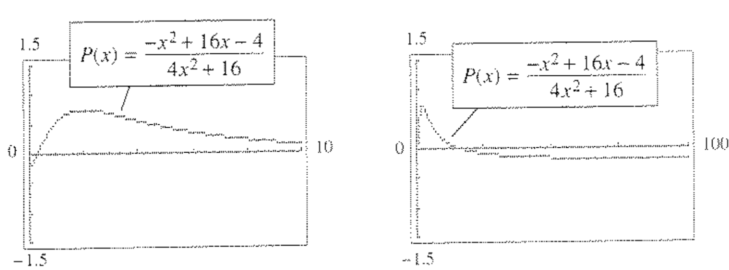


Figure 10.47 (a) (b)

• Checkpoint Solutions

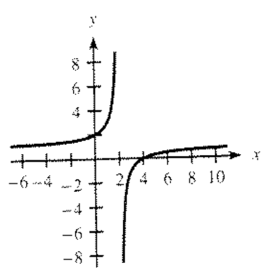
1. (a) True, $x = 1$ makes the denominator of $f(x)$ equal to zero, whereas the numerator is nonzero.
   (b) True, the function is rational with the degree of the numerator equal to that of the denominator and with the ratio of the leading coefficients equal to 2.
2. (a) False. There are no horizontal asymptotes, but $x = 0$ is a vertical asymptote.
   (b) True
   (c) True
   (d) True. The relative minimum point is $(-2, f(-2)) = (-2, 13)$.
   (e) True
   (f) False. There is a point of inflection only at $(\sqrt[3]{16}, 1)$. At $x = 0$ the vertical asymptote occurs, so there is no point on the graph and hence no point of inflection.

10.5 Exercises

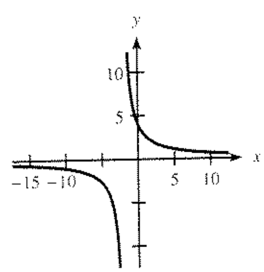
In Problems 1–4, a function and its graph are given. Use the graph to find each of the following, if they exist. Then confirm your results analytically.

(a) vertical asymptotes
(b) $\lim_{x\to\infty} f(x)$
(c) horizontal asymptotes
(d) $\lim_{x\to-\infty} f(x)$

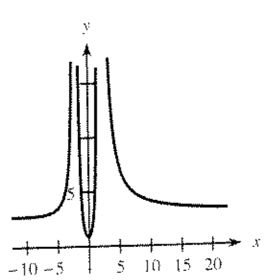
1. $f(x) = \dfrac{x-4}{x-2}$



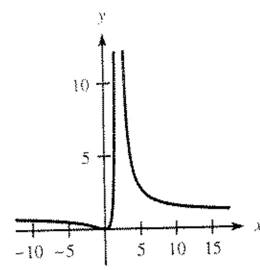
2. $f(x) = \dfrac{8}{x+2}$



3. $f(x) = \dfrac{3(x^4 + 2x^3 + 6x^2 + 2x + 5)}{(x^2-4)^2}$



4. $f(x) = \dfrac{x^2}{(x-2)^2}$



In Problems 5–10, find any horizontal and vertical asymptotes for each function.

5. $y = \dfrac{2x}{x-3}$
6. $y = \dfrac{3x-1}{x+5}$
7. $y = \dfrac{x+1}{x^2-4}$
8. $y = \dfrac{4x}{9-x^2}$
9. $y = \dfrac{3x^3-6}{x^2+4}$
10. $y = \dfrac{6x^3}{4x^2+9}$

For each function in Problems 11–18, find any horizontal and vertical asymptotes, and use information from the first derivative to sketch the graph.

11. $f(x) = \dfrac{2x+2}{x-3}$
12. $f(x) = \dfrac{5x-15}{x+2}$
13. $y = \dfrac{x^2+4}{x}$
14. $y = \dfrac{x^2+4}{x^2}$
15. $y = \dfrac{27x^2}{(x+1)^3}$
16. $y = \left(\dfrac{x+2}{x-3}\right)^2$
17. $f(x) = \dfrac{16x}{x^2+1}$
18. $f(x) = \dfrac{4x^2}{x^4+1}$

In Problems 19–24, a function and its first and second derivatives are given. Use these to find any horizontal and vertical asymptotes, critical points, relative maxima, relative minima, and points of inflection. Then sketch the graph of each function.

19. $y = \dfrac{x}{(x-1)^2}$

$y' = -\dfrac{x+1}{(x-1)^3}$

$y'' = \dfrac{2x+4}{(x-1)^4}$

20. $y = \dfrac{(x-1)^2}{x^2}$

$y' = \dfrac{2(x-1)}{x^3}$

$y'' = \dfrac{6-4x}{x^4}$

21. $y = x + \dfrac{3}{\sqrt[3]{x-3}}$

$y' = 1 - \dfrac{1}{(x-3)^{4/3}}$

$y'' = \dfrac{4}{3(x-3)^{7/3}}$

22. $y = 3\sqrt[3]{x} + \dfrac{1}{x}$

$y' = \dfrac{x^{4/3}-1}{x^2}$

$y'' = \dfrac{6-2x^{4/3}}{3x^3}$

23. $f(x) = \dfrac{9(x-2)^{2/3}}{x^2}$

$f'(x) = \dfrac{12(3-x)}{x^3(x-2)^{1/3}}$

$f''(x) = \dfrac{4(7x^2-42x+54)}{x^4(x-2)^{4/3}}$

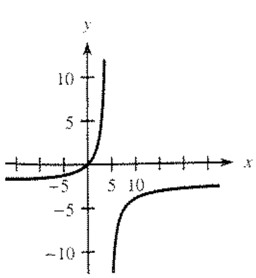
24. $f(x) = \dfrac{3x^{2/3}}{x+1}$

$f'(x) = \dfrac{2-x}{x^{1/3}(x+1)^2}$

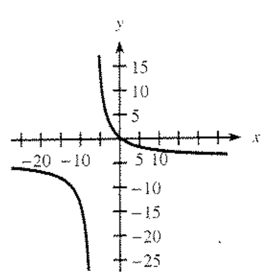
$f''(x) = \dfrac{2(2x^2-8x-1)}{3x^{4/3}(x+1)^3}$

In Problems 25–28, a function and its graph are given.
(a) Use the graph to estimate the locations of any horizontal or vertical asymptotes.
(b) Use the function to determine precisely the locations of any asymptotes.

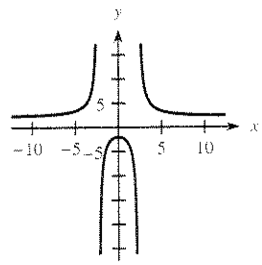
25. $f(x) = \dfrac{9x}{17-4x}$



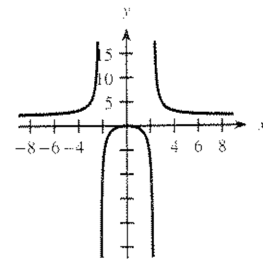
26. $f(x) = \dfrac{5-13x}{3x+20}$



27. $f(x) = \dfrac{20x^2+98}{9x^2-49}$



28. $f(x) = \dfrac{15x^2-x}{7x^2-35}$



For each function in Problems 29–34, complete the following steps.
(a) Use a graphing utility to graph the function in the standard viewing window.
(b) Analytically determine the location of any asymptotes and extrema.
(c) Graph the function in a viewing window that shows all features of the graph. State the ranges for x-values and y-values for your viewing window.

29. $f(x) = \dfrac{x+25}{x^2+1400}$

30. $f(x) = \dfrac{x-50}{x^2+1100}$

31. $f(x) = \dfrac{100(9-x^2)}{x^2+100}$

32. $f(x) = \dfrac{200x^2}{x^2+100}$

33. $f(x) = \dfrac{1000x-4000}{x^2-10x-2000}$

34. $f(x) = \dfrac{900x+5400}{x^2-30x-1800}$

APPLICATIONS

35. ***Cost-benefit*** The percent p of particulate pollution that can be removed from the smokestacks of an industrial plant by spending C dollars is given by

$$p = \frac{100C}{7300+C}$$

(a) Find any C-values at which the rate of change of p with respect to C does not exist. Make sure that these make sense in the problem.
(b) Find C-values for which p is increasing.
(c) If there is a horizontal asymptote, find it.
(d) Can 100% of the pollution be removed?

36. ***Cost-benefit*** The percent p of impurities that can be removed from the waste water of a manufacturing process at a cost of C dollars is given by

$$p = \frac{100C}{8100 + C}$$

(a) Find any C-values at which the rate of change of p with respect to C does not exist. Make sure that these make sense in the problem.
(b) Find C-values for which p is increasing.
(c) Find any horizontal asymptotes.
(d) Can 100% of the pollution be removed?

37. ***Revenue*** A recently released film has its weekly revenue given by

$$R(t) = \frac{50t}{t^2 + 36}, \qquad t \geq 0$$

where $R(t)$ is in millions of dollars and t is in weeks.
(a) Graph $R(t)$.
(b) When will revenue be maximized?
(c) Suppose that if revenue decreases for 4 consecutive weeks, the film will be removed from theaters and will be released as a video 12 weeks later. When will the video come out?

38. ***Minimizing average cost*** If the total daily cost, in dollars, of producing plastic rafts for swimming pools is given by

$$C(x) = 500 + 8x + 0.05x^2$$

where x is the number of rafts produced per day, then the average cost per raft produced is given by $\overline{C}(x) = C(x)/x$, for $x > 0$.
(a) Graph this function.
(b) Discuss what happens to the average cost as the number of rafts decreases, approaching 0.
(c) Find the level of production that minimizes average cost.

39. ***Wind chill*** If x is the wind speed in miles per hour and is greater than or equal to 5, then the wind chill (in degrees Fahrenheit) for an air temperature of 0°F can be approximated by the function

$$f(x) = \frac{289.173 - 58.5731x}{x + 1}, \qquad x \geq 5$$

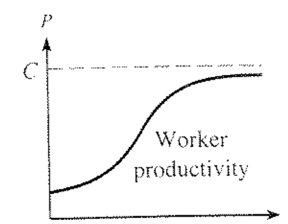
(a) Ignoring the restriction $x \geq 5$, does $f(x)$ have a vertical asymptote? If so, what is it?
(b) Does $f(x)$ have a vertical asymptote within its domain?
(c) Does $f(x)$ have a horizontal asymptote? If so, what is it?
(d) In the context of wind chill, does $\lim_{x\to\infty} f(x)$ have a physical interpretation? If so, what is it, and is it meaningful?

40. ***Profit*** An entrepreneur starts new companies and sells them when their growth is maximized. Suppose that the annual profit for a new company is given by

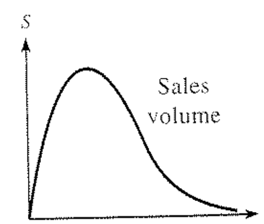
$$P(x) = 22 - \frac{1}{2}x - \frac{18}{x + 1}$$

where P is in thousands of dollars and x is the number of years after the company is formed. If she wants to sell the company before profits begin to decline, after how many years should she sell it?

41. ***Productivity*** The figure is a typical graph of worker productivity per hour P as a function of time t on the job.
(a) What is the horizontal asymptote?
(b) What is $\lim_{x\to\infty} P(t)$?
(c) What is the horizontal asymptote for $P'(t)$?
(d) What is $\lim_{x\to\infty} P'(t)$?



42. ***Sales volume*** The figure shows a typical curve that gives the volume of sales S as a function of time t after an ad campaign.
(a) What is the horizontal asymptote?
(b) What is $\lim_{t\to\infty} S(t)$?
(c) What is the horizontal asymptote for $S'(t)$?
(d) What is $\lim_{t\to\infty} S'(t)$?



Farm workers **The percent of U.S. workers in farm occupations during certain years is shown in the table.**

Year	Percent	Year	Percent	Year	Percent
1820	71.8	1930	21.2	1980	2.7
1850	63.7	1940	17.4	1985	2.8
1870	53.0	1950	11.6	1990	2.4
1900	37.5	1960	6.1	1994	2.5
1920	27.0	1970	3.6	2002	2.5

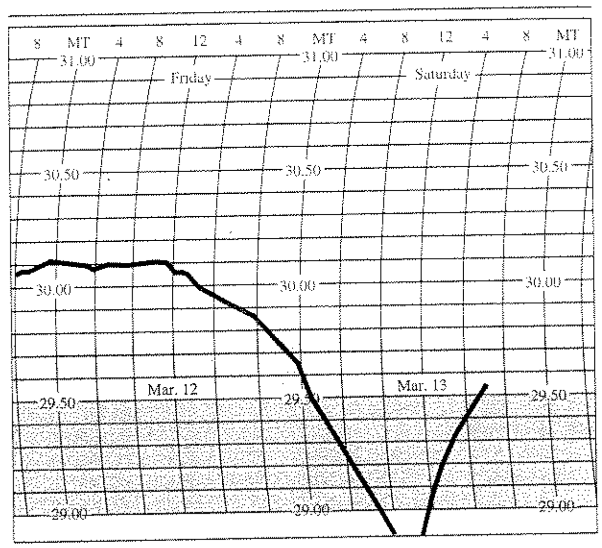
Source: Bureau of Labor Statistics, U.S. Department of Labor

Assume that the percent of U.S. workers in farm occupations can be modeled by the function

$$f(t) = 1000 \cdot \frac{-7.4812t + 1560.2}{1.2882t^2 - 122.18t + 21{,}483}$$

where t is the number of years past 1800. Use this model in Problems 43 and 44.

43. (a) Find $\lim_{t\to\infty} f(t)$.
    (b) Interpret your answer to part (a).
    (c) Does $f(t)$ have any vertical asymptotes within its domain $t \geq 0$?
44. (a) Use a graphing utility to graph $f(t)$ for $t = 0$ to $t = 220$.
    (b) From the graph, identify t-values where the model is inappropriate, and explain why it is inappropriate.
45. *Barometric pressure* The figure shows a barograph readout of the barometric pressure as recorded by Georgia Southern University's meteorological equipment. The figure shows a tremendous drop in barometric pressure on Saturday morning, March 13, 1993.
    (a) If $B(t)$ is barometric pressure expressed as a function of time, as shown in the figure, does $B(t)$ have a vertical asymptote sometime after 8 A.M. on Saturday, March 13, 1993? Explain why or why not.
    (b) Consult your library or some other resource to find out what happened in Georgia (and in the eastern United States) on March 13, 1993, to cause such a dramatic drop in barometric pressure.



Source: Statesboro Herald, March 14, 1993.

Key Terms and Formulas

Section	Key Terms	Formulas
10.1	Relative maxima and minima	
	Increasing	$f'(x) > 0$
	Decreasing	$f'(x) < 0$
	Critical points	$f'(x) = 0$ or $f'(x)$ undefined
	Sign diagram for $f'(x)$	
	First-derivative test	
	Horizontal point of inflection	
10.2	Concave up	$f''(x) > 0$
	Concave down	$f''(x) < 0$
	Point of inflection	May occur where $f''(x) = 0$ or $f''(x)$ undefined
	Sign diagram for $f''(x)$	
	Second-derivative test	

Section	Key Terms	Formulas
10.3	Absolute extrema Average cost Profit maximization Competitive market Monopolistic market	$\overline{C}(x) = C(x)/x$ $R(x) = p \cdot x$ where p = equilibrium price $R(x) = p \cdot x$ where $p = f(x)$ is the demand function
10.4	Inventory cost models	
10.5	Asymptotes Horizontal: $y = b$ Vertical: $x = c$ For rational function $y = f(x)/g(x)$	$f(x) \to b$ as $x \to \infty$ or $x \to -\infty$ y unbounded near $x = c$ Vertical: $x = c$ if $g(c) = 0$ and $f(c) \neq 0$ Horizontal: Key on highest-power terms of $f(x)$ and $g(x)$

Review Exercises

Section 10.1

In Problems 1–4, find all critical points and determine whether they are relative maxima, relative minima, or horizontal points of inflection.

1. $y = -x^2$
2. $p = q^2 - 4q - 5$
3. $f(x) = 1 - 3x + 3x^2 - x^3$
4. $f(x) = \dfrac{3x}{x^2 + 1}$

In Problems 5–10:
(a) Find all critical values, including those at which $f'(x)$ is undefined.
(b) Find the relative maxima and minima, if any exist.
(c) Find the horizontal points of inflection, if any exist.
(d) Sketch the graph.

5. $y = x^3 + x^2 - x - 1$
6. $f(x) = 4x^3 - x^4$
7. $f(x) = x^3 - \dfrac{15}{2}x^2 - 18x + \dfrac{3}{2}$
8. $y = 5x^7 - 7x^5 - 1$
9. $y = x^{2/3} - 1$
10. $y = x^{2/3}(x - 4)^2$

Section 10.2

11. Is the graph of $y = x^4 - 3x^3 + 2x - 1$ concave up or concave down at $x = 2$?
12. Find intervals on which the graph of $y = x^4 - 2x^3 - 12x^2 + 6$ is concave up and intervals on which it is concave down, and find points of inflection.
13. Find the relative maxima, relative minima, and points of inflection of the graph of $y = x^3 - 3x^2 - 9x + 10$.

In Problems 14 and 15, find any relative maxima, relative minima, and points of inflection, and sketch each graph.

14. $y = x^3 - 12x$
15. $y = 2 + 5x^3 - 3x^5$

Section 10.3

16. Given $R = 280x - x^2$, find the absolute maximum and minimum for R when (a) $0 \le x \le 200$ and (b) $0 \le x \le 100$.
17. Given $y = 6400x - 18x^2 - \dfrac{x^3}{3}$, find the absolute maximum and minimum for y when (a) $0 \le x \le 50$ and (b) $0 \le x \le 100$.

Section 10.5

In Problems 18 and 19, use the graphs to find the following items.
(a) vertical asymptotes
(b) horizontal asymptotes
(c) $\lim_{x \to +\infty} f(x)$
(d) $\lim_{x \to -\infty} f(x)$

18.