Axiomatic Probability

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- Probability Axioms:

  AXIOM 1 \( P(A) \geq 0 \).

  AXIOM 2 \( P(S) = 1 \).

  AXIOM 3 If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then

  \[ P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i) \]
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Proposition

$P(\emptyset) = 0$ where $\emptyset$ is the null event. This in turn implies that the property contained in Axiom 3 is valid for finite collection of events, i.e. if $A_1, A_2, \ldots, A_n$ is a finite collection of disjoint events, then $P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i)$
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For any event $A$, $P(A) + P(A') = 1$, from which $P(A) = 1 - P(A')$. 
Axiomatic Probability

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Proposition

For any two events $A$ and $B$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
Axiomatic Probability

Proposition

For any three events $A$, $B$, and $C$,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

$$- P(A \cap B) - P(B \cap C) - P(C \cap A)$$

$$+ P(A \cap B \cap C)$$
Determining Probabilities

In Probability, our main focus is to determine the probabilities for all possible events. However, some prior knowledge about the sample space is available. (While in Statistics, the prior knowledge is unavailable and we want to find it.)
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$$P(A) = \sum_{\text{all } E_i \text{'s that are in } A} P(E_i)$$
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Here the knowledge for $P(E_i)$ is given and we want to find $P(A)$. (In statistics, we want to find the knowledge about $P(E_i)$.)
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In that case, the probability \( p \) for each simple event \( E_i \) is determined by the size of the sample space \( N \), i.e.

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This is simply due to the fact

\[
1 = P(S) = P(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} P(E_i) = \sum_{i=1}^{N} p = N \cdot p
\]
Determining Probabilities

*Examples:*

tossing a fair coin: $N = 2$ and $P(\{\text{H}\}) = P(\{\text{T}\}) = \frac{1}{2};$

tossing a fair die: $N = 6$ and $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \frac{1}{6};$

randomly selecting a student from 25 students: $N = 25$ and $p = \frac{1}{25}.$
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Counting Techniques:
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If the sample space is finite and all the outcomes are equally likely to happen, then the formula

\[ P(A) = \sum_{\text{all } E_i \text{ s that are in } A} P(E_i) \]

simplifies to

\[ P(A) = \frac{N(A)}{N} \]

where \( E_i \) is any simple event, \( N \) is the number of outcomes of the sample space and \( N(A) \) is the number of outcomes contained in event \( A \).
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Determining the probability of \( A \) \( \Rightarrow \) counting \( N(A) \).
Determining Probabilities

Product Rule for Ordered Pairs:
Determining Probabilities

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Proposition

If the first element of object of an ordered pair can be selected in $n_1$ ways, and for each of these $n_1$ ways the second element of the pair can be selected in $n_2$ ways, then the number of pairs is $n_1 \cdot n_2$. 
Determining Probabilities

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Suppose a coin is tossed and then a marble is selected at random from a box containing one black (B), one red (R), and one green (G) marble. The possible outcomes are HB, HR, HG, TB, TR and TG. For each of the two possible outcomes of the coin there are three marbles that may be selected for a total of $2 \cdot 3 = 6$ possible outcomes.
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We can also use the tree diagram to illustrate:
Determining Probabilities

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Suppose we draw two cards from a deck of 52 cards. Each time, we record the suit of that card and then replace it. The outcome for each drawing is hearts (♥), diamonds (♦), clubs (♣) and spades (♠).
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If we are only interested in getting spades (♠), then there are 13 different outcomes for each drawing and the total number of outcomes with ♠♠ is $13 \cdot 13 = 169$. 
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If we are only interested in getting spades (♠), then there are 13 different outcomes for each drawing and the total number of outcomes with ♠♠ is $13 \cdot 13 = 169$.
However, if we do not replace the card after our first drawing, then there are still 13 outcomes with a ♠ for the first drawing but 12 outcomes with a ♠ for the second drawing. This time the total number of outcomes with ♠♠ is $13 \cdot 12 = 156$. 
Determining Probabilities

General Product Rule:

A $k$-tuple is an ordered collection of $k$ objects.

- e.g. $(2,3,1,6)$ from tossing a die 4 times;
- $(♣, ♠, ♥)$ from drawing cards; etc.

Proposition

Suppose a set consists of ordered collections of $k$ elements ($k$-tuples) and that there are $n_1$ possible choices for the first element; for each choice of the first element, there $n_2$ possible choices of the second element; ...; for each possible choice of the first $k-1$ elements, there are $n_k$ choices of the $k$th element. Then there are $n_1 \cdot n_2 \cdot \cdots \cdot n_k$ possible $k$-tuples.
Determining Probabilities

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Examples: We toss a coin 6 times. Then any outcome will be a 6-tuple. For example, (H,H,T,H,T,T). For each toss, we have two possibilities. Therefore the total number of outcomes would be

\[ 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64 \]
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If we restrict our attention to those outcomes starting with \(H\), then the total number of outcomes would be

\[
2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32
\]
Determining Probabilities

Permutations and Combinations

Example: Assume we are going to organize a club for learning SAS and all 25 students here will be the members. We need to elect someone to be the president, the vice-president and the treasurer. How many choices do we have?

If there is a university meeting and we need to find 3 members to present our activities, how many choices do we have to find the 3 representatives?

For the first question, the order is important while in the second question, the order doesn’t make any difference.
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Determining Probabilities

- **Permutation:** an ordered subset is called a permutation.

- **Combination:** an unordered subset is called a combination.

We use $P_k^n$ to denote the number of permutations of size $k$ that can be formed from $n$ individuals.

For example, selecting 3 digits from \{1, 2, 3, 4, 5, 6\} to form a 3-digit number; etc.

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Determining Probabilities

Proposition

\[ P_{k:n} = n \cdot (n - 1) \cdot \ldots \cdot (n - (k - 1)) = \frac{n!}{(n - k)!} \]

where \( k! = k \cdot (k - 1) \cdot \ldots \cdot 2 \cdot 1 \) is the \( k \) factorial.
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What is the probability that at least 3 of those selected are laser printers?
Determining Probabilities

Permutations and Combinations

Examples:

A photographer is going to take a photo for 9 students. 4 of the students are girls and the other 5 are boys. The photographer requires the students to stand in one row and girls should not be adjacent. How many choices could the photographer make?

If girls are allowed to be adjacent, but cannot form consecutive groups, how many choices could the photographer make?
Determining Probabilities

Permutations and Combinations

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A photographer is going to take a photo for 9 students. \( r \) of the students are girls and the other 5 are boys. The photographer requires the students to stand in one row and girls should not be adjacent. How many choices could the photographer make?

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\begin{array}{ccccccc}
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\end{array}
\]

If girls are allowed to be adjacent, but cannot form consecutive 3, how many choices could the photographer make?
Conditional Probability

Example 2.24

Complex components are assembled in a plant that uses two different assembly lines, A and B. Line A uses older equipment than B, so it is somewhat slower and less reliable. Suppose on a given day line A has assembled 8 components, of which 2 have been identified as defective (D) and 6 as nondefective (ND), whereas B has produced 1 defective and 9 nondefective components. This information is summarized in the following table.

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Unaware of this information, the sales manager randomly selects 1 of these 18 components for a demonstration.
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However, if the chosen component turns out to be defective, then it's more likely for the component to be produced by line A. That's because we are now focusing on column D and the component must have been 1 of 3 in the D column. Mathematically speaking, 

\[
P(A \mid D) = \frac{2}{3} = \frac{2}{18} = \frac{1}{9}
\]

\[
P(A \mid D) = \frac{N(A)}{N} = \frac{8}{18} = \frac{4}{9}
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P(\text{line A component selected}) = \frac{N(A)}{N} = \frac{8}{18} = \frac{4}{9}
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P(A \mid D) = \frac{2}{3} = \frac{\frac{2}{18}}{\frac{3}{18}} = \frac{P(A \cap D)}{PD}
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Because we have some prior knowledge that the selected component is defective.

\[ P(A) = \frac{4}{9} \implies P(A \mid D) = \frac{2}{3} \]
Conditional Probability

Definition

For any two events \(A\) and \(B\) with \(P(B) > 0\), the conditional probability of \(A\) given that \(B\) has occurred is defined by

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P(A | B) = \frac{P(A \cap B)}{P(B)}
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Event \(B\) is the prior knowledge. Due to the presence of event \(B\), the probability for event \(A\) to happen changed.
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Event $B$ is the *prior* knowledge. Due to the presence of event $B$, the probability for event $A$ to happen changed.
Conditional Probability

Example:
A maintenance firm has gathered the following information regarding the failure mechanisms for air conditioning systems:

- Evidence of gas leaks: Yes, No
- Evidence of electrical failure: Yes, No

The units without evidence of gas leaks or electrical failure showed other types of failure. If this is a representative sample of AC failure, find the probability for:

(a) there is evidence of electrical failure given that there was a gas leak;
(b) there is evidence of a gas leak given that there is evidence of electrical failure.
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<tr>
<th>evidence of gas leaks</th>
<th>yes</th>
<th>no</th>
</tr>
</thead>
<tbody>
<tr>
<td>yes</td>
<td>55</td>
<td>17</td>
</tr>
<tr>
<td>no</td>
<td>32</td>
<td>3</td>
</tr>
</tbody>
</table>

The units without evidence of gas leaks or electrical failure showed other types of failure. If this is a representative sample of AC failure, find the probability for

(a) there is evidence of electrical failure given that there was a gas leak;

(b) there is evidence of a gas leak given that there is evidence of electrical failure.
Conditional Probability

The Multiplication Rule

\[ P(A \cap B) = P(A | B) \cdot P(B) \]

This is obtained directly from the definition of conditional probability:

\[ P(A | B) = \frac{P(A \cap B)}{P(B)} \]

Sometimes, we know the probability for event A to happen conditioned on the presence of event B, then we can use the multiplication rule to calculate the probability that event A and B happening simultaneously.
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Example 2.27 Four individuals have responded to a request by a blood bank for blood donations. None of them has donated before, so their blood types are unknown. Suppose only type O+ is desired and only one of the four actually has this type. If the potential donors are selected in random order for typing, what is the probability that at least three individuals must be typed to obtain the desired type?

What is the probability that exactly three individuals are typed to obtain the desired type?

Remark: the multiplication rule can be extended to multiple events:

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 | A_1 \cap A_2) \cdot P(A_1 \cap A_2)$$
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Example 2.29 A chain of video stores sells three different brands of DVD players. Of its DVD player sales, 50% are brand 1, 30% are brand 2, and 20% are brand 3. Each manufacturer offers a 1-year warranty on parts and labor. It is known that 25% of brand 1’s DVD players require warranty on parts and labor, whereas the corresponding percentages for brands 2 and 3 are 20% and 10%, respectively.

1. What is the probability that a randomly selected purchaser has bought a brand 1 DVD player that will need repair while under warranty?

2. What is the probability that a randomly selected purchaser has a DVD player that will need repair while under warranty?
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