Real & Complex Analysis Qualifying
Exam Solution, Spring 2010

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A-1

(a) Given $\epsilon > 0$, $P(|f_n - f_m| > \epsilon) < \frac{1}{\epsilon^p} \int_0^1 |f_n(x) - f_m(x)|^p \, dx$. Therefore, for each $j \in \mathbb{N}$, there exists $N_j$ such that $P(|f_n - f_m| > \frac{1}{2^j}) < \frac{1}{2^j}$ for every $n, m \geq N_j$. If we choose $n = N_j, m = N_j + 1$, and define $A_j := \{f_{N_j} \neq f_{N_j + 1}\}$, we find that $\{f_{N_j}(x)\}$ is Cauchy on $(\bigcup_{m=n}^\infty A_m)^c$ for every $n$; thus $\{g_j(x) := f_{N_j}(x) \rightarrow f(x)\}$ for some finite a.e. $f$ and the proof is complete.

(b) Let $f_1 = \chi_{[0,1/2]}, f_2 = \chi_{[1/2,1]}, f_3 = \chi_{[0,1/4]}, f_4 = \chi_{[1/4,2/4]}, f_5 = \chi_{[2/4,3/4]}, f_6 = \chi_{[3/4,1]}, f_7 = \chi_{[0,1/8]}, f_8 = \chi_{[1/8,2/8]}, \ldots$, and so on.

A-2

(a) Approach 1. We would show that the inclusion map $i$ is a closed operator. If $f_n \rightarrow f$ in $L^p(X)$, and $i(f_n) = f_n \rightarrow g$ in $L^q(X)$, then by the application of Chebyshev’s inequality we have $f_n \rightarrow f$ in measure and $f_n \rightarrow g$ in measure. Thus, $f = g$ a.e. $X$. By closed graph theorem, The inclusion $i$ is continuous linear operator and hence it is bounded.

Approach 2. The idea is basically due to Shiang Tang. Assume that $i$ is not bounded. Thus, there exists $\|f_n\|_p = 1$ for all $n$ but $\|f_n\|_p \geq n^{2^n}$. Consider $f = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n|$. It is obvious that $f \in L^p(X)$; however, for every $n \in \mathbb{N}$, $\|f\|_q \geq \|\frac{1}{2^n} |f_n|\|_q \geq n$, so $f \notin L^q(X)$, contradicts the fact that $L^p(X) \subset L^2(X)$.

(b) Assume that $\inf_{E' \in M'} \mu(E') = 0$, so we could pick a sequence of $A_n \in M'$ so that $\mu(A_n) \rightarrow 0$. Let $f_n = \mu(A_n)^{-1/p} \chi_{A_n}$, and it is easy to
check that \( \| f_n \|_p = 1 \) for every \( n \in \mathbb{N} \). However, \( \| f_n \|_q = \mu(A_n)^{1/q - 1/p} \to \infty \) as \( n \to \infty \). (a) above is thus violated.

Remarks. To prove conversely, consider \( \inf_{E' \in M'} \mu(E') = m > 0 \). For every \( f \in L^p \), we claim that \( f \) is bounded and hence \( f \in L^q \). Otherwise, 
\[
\infty > \int_X |f|^p \, d\mu = \sum_{n=0}^{\infty} \int_X |f|^p \chi_{\{n \leq |f| < n+1\}} \, d\mu \geq \sum_{n=1}^{\infty} n^p \mu(n \leq |f| < n + 1) \geq \sum_{n=1}^{\infty} \mu(n \leq |f| < n + 1) = \infty,
\]
since unboundedness of \( f \) will result in infinitely many positive terms that are larger than \( m \) in \( \sum_{n=1}^{\infty} \mu(n \leq |f| < n + 1) \), and this is a contradiction.

A-3

(a) Decompose \( H = V \oplus V^\perp \). For every \( h = v + w, h' = v' + w' \), where \( v, v' \in V \) and \( w, w' \in V^\perp \),

(i) \( \pi_V(h) = \pi_V^2(h) = v \) - idempotent

(ii) Since \( \|h\|^2 = \|v\|^2 + \|w\|^2 \), \( \frac{\|\pi_V(h)\|}{\|h\|} = \frac{\|v\|}{\|h\|} \leq 1 \) for all \( h \in H \). – Norm less than 1.

(iii) \( (\pi_V(h), h') = (v, v' + w') = (v + w, v') = (h, \pi_V(h')) \). – Self adjoint

(b) Let \( N(P) \) be the null space of \( P \). we claim that \( N(P) \) is closed, for if \( a_n \in N(P) \) and \( a_n \to a \in N(P), P(a_n) = 0 \to P(a) = 0 \) by continuity of \( P \).

Now we decompose \( H = N(P) \oplus N(P)^\perp \). For every \( a \in N(P), b \in N(P)^\perp, 0 = (P(a), b) = (a, P(b)) \), hence \( P(b) \in N(P)^\perp \) for all \( b \in N(P)^\perp \).

We claim that \( b = P(b) \) for all \( b \in N(P)^\perp \). Consider \( P(b - P(b)) = P(b) - P^2(b) = P(b) - P(b) = 0 \), showing \( b - P(b) \in N(P) \). However, \( b - P(b) \) also belongs to \( N(P)^\perp \) due to the arguments above. The only possibility is that \( b - P(b) = 0 \).

Therefore, for every \( u = a + b \in H, a \in N(P) \) and \( b \in N(P)^\perp, P(a + b) = P(b) = b \), and the proof is complete.

A-4

(a) Norm-preserving: For every \( \xi \in l^\infty, \|\xi\|_{l^\infty} = 1 \), \( \Lambda_\eta(\xi) \leq \sum_i |\eta_i| = \|\eta\|_\mu \). Thus \( \|\eta\|_{(l^\infty)^*} \leq \|\eta\|_\mu \). Conversely, if we choose \( \xi \) so that \( \xi_i = \text{sgn}(\eta_i) \),
then $\|\eta\|_{(l^\infty)^*} \geq \Lambda_\eta(\xi) = \sum_i |\eta_i| = \|\eta\|_\mu$.

Injection: if $\sum_i \eta_i \xi_i = \sum_i \eta'_i \xi_i$ for every $\xi_i \in l^\infty$, then we may choose $\xi$ so that $\xi_j = 1$ and $\xi_k = 0$ for $k \neq j$. Thus $\eta_i = \eta'_i$ for every $i$ and hence $\eta = \eta'$.

Not onto: let $c \in (l^\infty)^*$ be such that $c(\xi) = \limsup_n \xi_n$ and assume that $c = \Lambda_\eta$ for some $\eta \in l^1$. If we choose $\xi$ such that $\xi_k$ vanishes in every subscript $k$ but some $j$, then we have $\eta_j = 0$. Arbitrariness of $j$ shows that $\eta = 0$ and $c$ is a zero linear functional, which is a contradiction.

(b) $\mu : \sigma$–finite positive measure; $1 \leq q < \infty$; $p$ is chosen s.t. $\frac{1}{p} + \frac{1}{q} = 1$. For a proof, see Rudin page 127.

A-5 (Poincare’s recurrence theorem)

Given $E \in M$, let the collection of sets satisfying this property be $A = A_E$, and note that $A$ is measurable. We claim that $\{T^{-n}(A)\}_n$ are pairwise disjoint. If, $x \in T^{-n}(A) \cap T^{-m}(A), m > n$, then $T^n(x) \in A$ and $T^{-m}T^n(x) \in A$, a contradiction to the definition of $A$.

Therefore, $\infty > \mu(X) \geq \mu(\bigcup_{n=1}^\infty T^{-n}(A)) = \sum_{n=1}^\infty \mu(T^{-n}(A)) = \sum_{n=1}^\infty \mu(A)$, the only possibility is that $\mu(A) = 0$.

B-6

(a) $Res(f;1) = \lim_{z \to 1} (z - 1) f(z) = \frac{6+2}{2^2} = 1$; $Res(f;-1) = \lim_{z \to -1} (z + 1) f(z) = \frac{-6+2}{2^2} = 1$; by the residue theorem, $\int \gamma f(z) \, dz = 2\pi i (Res(f;1) + Res(f;-1)) = 4\pi i$. 3
(b) 

\[ f(z) = (6z + 2) \cdot \frac{1}{3z^2} \cdot (1 - \frac{1}{z^2})^{-1} \cdot (1 + \frac{z}{3})^{-1} \]

\[ = (6z + 2) \cdot \frac{1}{3z^2} \cdot \sum_{n=0}^{\infty} \frac{1}{z^{2n}} \sum_{m=0}^{\infty} (-1)^m z^m \cdot \frac{3}{3m} \]

\[ = \sum_{n,m=0}^{\infty} \left( 2 \cdot (-1)^m \frac{1}{z^{2n}} \cdot \frac{3}{3m} + \frac{2}{3} (-1)^m \frac{1}{z^{2n}} \cdot \frac{3}{3m} \right) \]

\[ = \sum_{k=-\infty}^{\infty} \left( \sum_{m,n\geq0; m-1-2n=k} \frac{2(-1)^m}{3m} + \sum_{m,n\geq0; m-2n=k} \frac{2(-1)^m}{3m+1} \right) z^k. \]

Where the interchange of summation is due to absolute and uniform convergence. The coefficient of \( z^{-1} \) is \( \sum_{n=0}^{\infty} \frac{2(-1)^{2n}}{32n} + \sum_{n=0}^{\infty} \frac{2(-1)^{2n+1}}{32n+2} = 2 \). Therefore, \( \int_{\gamma} f(z) \, dz = \int_{\gamma} \frac{2}{z} \, dz = 4\pi i. \)

B-7

\[
\int_{0}^{2\pi} \cos^{2n} \theta \, d\theta = \int_{|z|=1} \frac{1}{i} (\frac{1}{2} (z + z^{-1}))^{2n} \, \frac{1}{iz} \, dz \\
= \frac{1}{i4^n} \int_{|z|=1} \frac{(z^2 + 1)^{2n}}{z^{2n+1}} \, dz \\
= 2\pi i \cdot \frac{1}{i4^n} \text{Res} \left( \frac{(z^2 + 1)^{2n}}{z^{2n+1}} ; 0 \right) = \frac{2\pi}{4^n} \cdot \left( \frac{2n}{n} \right).
\]

B-8

(a) Let \( f'(z_0) = 0 \). If \( f'(z) \equiv 0 \) on \( U \), then \( f \) is a constant, and of course it is not injective. So by uniqueness theorem we may assume that there is some neighborhood \( B_{z_0}(R) \) of \( z_0 \) such that \( f'(z) \neq 0 \) for \( z \in B_{z_0}(R) \setminus \{0\} \).

Since \( f'(z_0) = 0 \), we have \( f(z) = f(z_0) + (z-z_0)^k g(z) \) in a neighborhood \( B := B_{z_0}(r) \) of \( z_0 \), where \( k \geq 2 \), and \( g \) is holomorphic and non-vanishing on \( B \). And here we choose \( r < R \). Let \( m := \frac{1}{2} \sup_{\xi \in \partial B} |(z-z_0)^k g(z)| > 0 \), we have that \( (z-z_0)^k g(z) + m \) and \( (z-z_0)^k g(z) \) have both \( k \) zeros on \( B \) by
Rouche’s theorem. Therefore, \( f(z) - f(0) + m \) has \( k \) zeros on \( B \).

Since \( (f(z) - f(0) + m)' = f'(z) \neq 0 \) on \( B \setminus \{0\} \), These \( k \) zeros \( \{z_1, \cdots, z_k\} \) of \( f(z) - f(0) + m \) on \( B \) are all distinct, and obviously none of them equals 0. Therefore, \( f(z_1) = \cdots = f(z_k) \), thus \( f \) is not injective.

(b) False. Let \( f(z) = z^2 \) defined on an open connected set which contains \( \{1, -1\} \) but does not contain \( \{0\} \).

**B-9**

Consider \( a(z) = \frac{2+i z}{2-iz} \), which is analytic on \( HD_2 := \{ z : |z| < 2, \, \text{Re} \, z > 0 \} \), and it maps \( HD_2 \) to \( Q := \{ z = a + bi : a > 0, b > 0 \} \). To see this, notice that \( \text{Arg}(2+z) = \text{Arg}(z-(-2)) \), and if we pick \(-2,2\), and \( z \in HD_2 \) as vertices of a triangle, it is an obtuse triangle. Therefore, \( 0 < \text{Arg}(2+z) - \text{Arg}(2-z) < \frac{\pi}{2} \), and this shows \( \frac{2+i z}{2-iz} \in Q \). Besides, for any \( z \in Q \), \( z = \frac{2+w}{2-w} \Rightarrow w = \frac{2z-2}{z+1} \in HD_2 \), we may define \( a^{-1}(z) : Q \to HD_2 \) by \( a^{-1}(z) = \frac{2z-2}{z+1} \), which is the inverse map of \( a \). Therefore, \( a \) is a conformal map (analytic and bijective) from \( HD_2 \) to \( Q \).

Let \( b(z) = z^2 : Q \to H := \{ z : \text{Im} \, z > 0 \} \). It is easy to check that \( b \) is also a conformal map.

Finally, we define \( c(z) = \frac{i - z}{i + z} \), which is analytic on \( H \), and for \( z = a + bi \), \( b > 0 \), \( |i - z| = | - a + (1 - b)i| < |a + (1 + b)i| = |i + z| \), which shows that \( c : H \to D := \{ z : |z| < 1 \} \). To see \( c(z) \) is onto and \( 1 \rightarrow 1 \), for any \( w = re^{i\theta} \in D \), \( r < 1 \), \( w = \frac{i - z}{i + z} \Rightarrow z = \frac{i - re^{i\theta}}{1 + re^{i\theta}} = \frac{i - r^2 - 2i \sin(\theta)}{1 + r^2 + 2r \cos(\theta)} \in H \). Thus, we may define the inverse map of \( c \) by \( c^{-1}(z) = i \frac{1 - z}{1 + z} : D \to H \). These facts show that \( c \) is a conformal map from \( H \) to \( D \).

Therefore, \( F(z) := (c \circ b \circ a)(z) = \frac{i - (\frac{2 + iz}{2 - iz})^2}{1 + (\frac{2 + iz}{2 - iz})^2} \) is a conformal map from \( HD_2 \) to \( D \). If \( \phi(z) \) is another conformal map from \( HD_2 \) to \( D \), then \( \phi \circ F^{-1} \) is a conformal map from \( D \) to \( D \), which takes form \( e^{i\theta} \frac{a - z}{1 - az} \) for some \( a \in D \). As a result, \( \phi(z) = e^{i\theta} \frac{a - F(z)}{1 - aF(z)} \).
Define $g(\zeta) = \frac{f(\zeta) - f(w)}{\zeta - w}$, $g(w) = f'(w)$. Since $g$ is continuous on the unit disk $D$, and is holomorphic on $D \setminus \{0\}$, thus $g$ is holomorphic on $D$ by Morera’s theorem.

As a result, when $z \neq w$, we have $|g(w)| \leq \sup_{\zeta \in \partial B_0(\frac{1+r}{1-r})} |g(\zeta)| \leq \frac{2M}{1+r^2}$. That is, $|f(z) - f(w)| \leq \frac{4M}{1+r^2}|z - w|$, and the constant is independent of our choices of $z, w$. 