Real & Complex Analysis Qualifying Exam Solution, Spring 2010

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A-1

(a) Given $\epsilon > 0$, $P(|f_n - f_m| > \epsilon) < \frac{1}{\epsilon^p}\int_0^1 |f_n(x) - f_m(x)|^p\,dx$. Therefore, for each $j \in \mathbb{N}$, there exists N_j such that $P(|f_n - f_m| > \frac{1}{2^j}) < \frac{1}{2^j}$ for every $n, m \geq N_j$. If we choose $n = N_j, m = N_{j+1}$, and define $A_j := \{|f_{N_j} - f_{N_{j+1}}| > \frac{1}{2^j}\}$, we find that $\{f_{N_j}(x)\}$ is Cauchy on $(\bigcup_{m=n}^{\infty} A_m)^c$ for every n; thus $\{f_{N_j}(x)\}$ is Cauchy a.e. x, actually. As a result,$g_j(x) := f_{N_j}(x) \to f(x)$ for some finite a.e. f and the proof is complete.

(b)Let $f_1 = \chi_{[0,1/2]}$, $f_2 = \chi_{[1/2,1]}$, $f_3 = \chi_{[0,1/4]}$, $f_4 = \chi_{[1/4,2/4]}$, $f_5 = \chi_{[2/4,3/4]}$, $f_6 = \chi_{[3/4,1]}$, $f_7 = \chi_{[0,1/8]}$, $f_8 = \chi_{[1/8,2/8]}$, $\cdots$, and so on.

A-2

(a)**Approach 1.** We would show that the inclusion map i is a closed operator. If $f_n \to f$ in $L^p(X)$, and $i(f_n) = f_n \to g$ in $L^q(X)$, then by the application of Chebeshev's inequality we have $f_n \to f$ in measure and $f_n \to g$ in measure. Thus, $f = g$ a.e. X. By closed graph theorem, The inclusion i is continuous linear operator and hence it is bounded.

Approach 2. The idea is basically due to Shiang Tang. Assume that i is not bounded. Thus, there exists $\|f_n\|_p = 1$ for all n but $\|f_n\|_p \geq n2^n$. Consider $f = \sum_{n=1} \frac{1}{2^n}|f_n|$. It is obvious that $f \in L^p(X)$; however, for every $n \in \mathbb{N}$, $\|f\|_q \geq \|\frac{1}{2^n}f_n\|_q \geq n$, so $f \notin L^q(X)$, contradicts the fact that $L^p(X) \subset L^q(X)$.

(b) Assume that $\inf_{E' \in M'} \mu(E') = 0$, so we could pick a sequence of $A_n \in M'$ so that $\mu(A_n) \to 0$. Let $f_n = \mu(A_n)^{-1/p}\chi_{A_n}$, and it is easy to

check that $\|f_n\|_p = 1$ for every $n \in \mathbb{N}$. However, $\|f_n\|_q = \mu(A_n)^{1/q-1/p} \to \infty$ as $n \to \infty$. (a) above is thus violated.

Remarks. To prove conversely, consider $\inf_{E' \in M'} \mu(E') = m > 0$. For every $f \in L^p$, we claim that f is bounded and hence $f \in L^q$. Otherwise, $\infty > \int_X |f|^p \, d\mu = \sum_{n=0}^{\infty} \int_X |f|^p \chi_{\{n \le |f| < n+1\}} \, d\mu \ge \sum_{n=0}^{\infty} n^p \mu(n \le |f| < n+1) \ge \sum_{n=1}^{\infty} \mu(n \le |f| < n+1) = \infty$, since unboundedness of f will result in infinitely many positive terms that are larger than m in $\sum_{n=1}^{\infty} \mu(n \le |f| < n+1)$, and this is a contradiction.

A-3

(a) Decompose $H = V \oplus V^{\perp}$. For every $h = v + w, h' = v' + w'$, where $v, v' \in V$ and $w, w' \in V^{\perp}$,

(i)$\pi_V(h) = \pi_V^2(h) = v$ – idempotent
(ii) Since $\|h\|^2 = \|v\|^2 + \|w^2\|$, $\frac{\|\pi_V(h)\|}{\|h\|} = \frac{\|v\|}{\|h\|} \le 1$ for all $h \in H$. – Norm less than 1.
(iii) $(\pi_V(h), h') = (v, v' + w') = (v + w, v') = (h, \pi_V(h'))$. – Self adjoint

(b) Let $N(P)$ be the null space of P. we claim that $N(P)$ is closed, for if $a_n \in N(P)$ and $a_n \to a \in N(P)$, $P(a_n) = 0 \to P(a) = 0$ by continuity of P.

Now we decompose $H = N(P) \oplus N(P)^{\perp}$. For every $a \in N(P)$, $b \in N(P)^{\perp}$, $0 = (P(a), b) = (a, P(b))$, hence $P(b) \in N(P)^{\perp}$ for all $b \in N(P)^{\perp}$.

We claim that $b = P(b)$ for all $b \in N(P)^{\perp}$. Consider $P(b - P(b)) = P(b) - P^2(b) = P(b) - P(b) = 0$, showing $b - P(b) \in N(P)$. However, $b - P(b)$ also belongs to $N(P)^{\perp}$ due to the arguments above. The only possibility is that $b - P(b) = 0$.

Therefore, for every $u = a + b \in H$, $a \in N(P)$ and $b \in N(P)^{\perp}$, $P(a+b) = P(b) = b$, and the proof is complete.

A-4

(a) Norm-preserving: For every $\xi \in l^{\infty}, \|\xi\|_{l^{\infty}} = 1$, $\Lambda_{\eta}(\xi) \le \sum_i |\eta_i| = \|\eta\|_{l^1}$. Thus $\|\eta\|_{(l^{\infty})^*} \le \|\eta\|_{l^1}$. Conversely, if we choose ξ so that $\xi_i = sgn(\eta_i)$,

then $\|\eta\|_{(l^\infty)^*} \geq \Lambda_\eta(\xi) = \sum_i |\eta_i| = \|\eta\|_{l^1}$.

Injection: if $\sum_i \eta_i \xi_i = \sum_i \eta_i' \xi_i$ for every $\xi_i \in l^\infty$, then we may choose ξ so that $\xi_j = 1$ and $\xi_k = 0$ for $k \neq j$. Thus $\eta_i = \eta_i'$ for every i and hence $\eta = \eta'$.

Not onto: let $c \in (l^\infty)^*$ be such that $c(\xi) = \limsup_n \xi_n$ and assume that $c = \Lambda_\eta$ for some $\eta \in l^1$. If we choose ξ such that ξ_k vanishes in every subscript k but some j, then we have $\eta_j = 0$. Arbitrariness of j shows that $\eta = 0$ and c is a zero linear functional, which is a contradiction.

(b)$\mu : \sigma-$finite positive measure; $1 \leq q < \infty$; p is chosen s.t. $\frac{1}{p} + \frac{1}{q} = 1$. For a proof, see Rudin page 127.

A-5 (Poincare's recurrence theorem)

Given $E \in M$, let the collection of sets satisfying this property be $A = A_E$, and note that A is measurable. We claim that $\{T^{-n}(A)\}_n$ are pairwise disjoint. If, $x \in T^{-n}(A) \cap T^{-m}(A), m > n$, then $T^n(x) \in A$ and $T^{m-n}T^n(x) \in A$, a contradiction to the definition of A.

Therefore, $\infty > \mu(X) \geq \mu(\bigcup_{n=1}^\infty T^{-n}(A)) = \sum_{n=1}^\infty \mu(T^{-n}(A)) = \sum_{n=1}^\infty \mu(A)$, the only possibility is that $\mu(A) = 0$.

B-6

(a) $Res(f;1) = \lim_{z\to 1}(z-1)f(z) = \frac{6+2}{2\cdot 4} = 1$; $Res(f;-1) = \lim_{z\to -1}(z+1)f(z) = \frac{-6+2}{-2\cdot 2} = 1$; by the residue theorem, $\int_\gamma f(z)\,dz = 2\pi i(Res(f;1) + Res(f;-1)) = 4\pi i$.

(b)

$$
\begin{aligned}
f(z) &= (6z+2)\cdot\frac{1}{3z^2}\cdot(1-\frac{1}{z^2})^{-1}\cdot(1+\frac{z}{3})^{-1}\\
&= (6z+2)\cdot\frac{1}{3z^2}\cdot\sum_{n=0}^{\infty}\frac{1}{z^{2n}}\sum_{m=0}^{\infty}(-1)^m\frac{z^m}{3^m}\\
&= \sum_{n,m=0}^{\infty}\big(2\cdot(-1)^m\frac{1}{z^{2n}}\frac{z^{m-1}}{3^m}+\frac{2}{3}(-1)^m\frac{1}{z^{2n}}\frac{z^{m-2}}{3^m}\big)\\
&= \sum_{k=-\infty}^{\infty}\Big(\sum_{m,n\geq 0:m-1-2n=k}\frac{2(-1)^m}{3^m}+\sum_{m,n\geq 0:m-2-2n=k}\frac{2(-1)^m}{3^{m+1}}\Big)z^k.
\end{aligned}
$$

Where the interchange of summation is due to absolute and uniform convergence. The coefficient of z^{-1} is $\sum_{n=0}^{\infty}\frac{2(-1)^{2n}}{3^{2n}}+\sum_{n=0}^{\infty}\frac{2(-1)^{2n+1}}{3^{2n+2}}=2$. Therefore, $\int_\gamma f(z)\,dz=\int_\gamma\frac{2}{z}\,dz=4\pi i$.

B-7

$$
\begin{aligned}
\int_0^{2\pi}\cos^{2n}\theta\,d\theta &= \int_{|z|=1}(\frac{1}{2}(z+z^{-1}))^{2n}\frac{1}{iz}\,dz\\
&= \frac{1}{i4^n}\int_{|z|=1}\frac{(z^2+1)^{2n}}{z^{2n+1}}\,dz\\
&= 2\pi i\cdot\frac{1}{i4^n}Res(\frac{(z^2+1)^{2n}}{z^{2n+1}};0)=\frac{2\pi}{4^n}\cdot\binom{2n}{n}.
\end{aligned}
$$

B-8

(a) Let $f'(z_0)=0$. If $f'(z)\equiv 0$ on U, then f is a constant, and of course it is not injective. So by uniqueness theorem we may assume that there is some neighborhood $B_{z_0}(R)$ of z_0 such that $f'(z)\neq 0$ for $z\in B_{z_0}(R)\setminus\{0\}$.

Since $f'(z_0)=0$, we have $f(z)=f(z_0)+(z-z_0)^k g(z)$ in a neighborhood $B:=B_{z_0}(r)$ of z_0, where $k\geq 2$, and g is holomorphic and non-vanishing on B. And here we choose $r<R$. Let $m:=\frac{1}{2}\sup_{\xi\in\partial B}|(z-z_0)^k g(z)|>0$, we have that $(z-z_0)^k g(z)+m$ and $(z-z_0)^k g(z)$ have both k zeros on B by

Rouche's theorem. Therefore, $f(z) - f(0) + m$ has k zeros on B.

Since $(f(z)-f(0)+m)' = f'(z) \neq 0$ on $B\backslash\{0\}$, These k zeros $\{z_1, \cdots, z_k\}$ of $f(z) - f(0) + m$ on B are all distinct, and obviously none of them equals 0. Therefore, $f(z_1) = \cdots = f(z_k)$, thus f is not injective.

(b) False. Let $f(z) = z^2$ defined on an open connected set which contains $\{1, -1\}$ but does not contain $\{0\}$.

B-9

Consider $a(z) = \frac{2+z}{2-z}$, which is analytic on $HD_2 := \{z : |z| < 2, Re\, z > 0\}$, and it maps HD_2 to $Q := \{z = a+bi : a > 0, b > 0\}$. To see this, notice that $Arg(2+z) = Arg(z-(-2))$, and if we pick -2,2, and $z \in HD_2$ as vertices of a triangle, it is an obtuse triangle. Therefore, $0 < Arg(2+z) - Arg(2-z) < \frac{\pi}{2}$, and this shows $\frac{2+z}{2-z} \in Q$. Besides, for any $z \in Q$, $z = \frac{2+w}{2-w} \Rightarrow w = \frac{2z-2}{z+1} \in HD_2$, we may define $a^{-1}(z) : Q \to HD_2$ by $a^{-1}(z) = \frac{2z-2}{z+1}$, which is the inverse map of a. Therefore, a is a conformal map (analytic and bijective) from HD_2 to Q.

Let $b(z) = z^2 : Q \to H := \{z : Im\, z > 0\}$. It is easy to check that b is also a conformal map.

Finally, we define $c(z) = \frac{i-z}{i+z}$, which is analytic on H, and for $z = a + bi$, $b > 0$, $|i - z| = |-a + (1 - b)i| < |a + (1 + b)i| = |i + z|$, which shows that $c : H \to D := \{z : |z| < 1\}$. To see $c(z)$ is onto and $1 - 1$, for any $w = re^{i\theta} \in D$, $r < 1$, $w = \frac{i-z}{i+z} \Rightarrow z = i\frac{1-re^{i\theta}}{1+re^{i\theta}} = i\frac{1-r^2-2i\sin(\theta)}{1+r^2+2r\cos(\theta)} \in H$. Thus, we may define the inverse map of c by $c^{-1}(z) = i\frac{1-z}{1+z} : D \to H$. These facts show that c is a conformal map from H to D.

Therefore, $F(z) := (c \circ b \circ a)(z) = \frac{i-(\frac{2+z}{2-z})^2}{i+(\frac{2+z}{2-z})^2}$ is a conformal map from HD_2 to D. If $\phi(z)$ is another conformal map from HD_2 to D, then $\phi \circ F^{-1}$ is a conformal map from D to D, which takes form $e^{i\theta}\frac{a-z}{1-\bar{a}z}$ for some $a \in D$. As a result, $\phi(z) = e^{i\theta}\frac{a-F(z)}{1-\bar{a}F(z)}$.

B-10

Define $g(\zeta) = \frac{f(\zeta)-f(w)}{\zeta-w}$, $g(w) = f'(w)$. Since g is continuous on the unit disk D, and is holomorphic on $D \setminus \{0\}$, thus g is holomorphic on D by Morera's theorem.

As a result, when $z \neq w$, we have $|g(w)| \leq \sup_{\zeta \in \partial B_0(\frac{1+r}{2})} |g(\zeta)| \leq \frac{2M}{(1+r)/2}$. That is, $|f(z) - f(w)| \leq \frac{4M}{1+r}|z-w|$, and the constant is independent of our choices of z, w.