Real & Complex Analysis Qualifying Exam Solution, Fall 2010

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A-1

Note that $\{x : f(x) > a\} = \cap_{b\in\mathbb{Q},b<a}\{x : f(x) > b\}$ is measurable, $\{x : f(x) < a\} = \cap_{b\in\mathbb{Q},b>a}\{x : f(x) < b\}$ is measurable, and $\{x : f(x) \in O\}$ is measurable for every open set O in $\mathbb{R}$ since O may be decomposed into a countable union of open intervals.

Also, $F := \{A : f^{-1}(A) \text{ is measurable }\}$ is a sigma-algebra containing all open sets in $\mathbb{R}$. It means that $\mathscr{B}(\mathbb{R}) \subseteq F$. Thus, for every $B \in \mathscr{B}(\mathbb{R})$, $f^{-1}(B)$ is measurable and we're done.

A-2

(1)

$$\begin{aligned}\|f_ng_n - fg\|_{L^1} &\leq \|f_ng_n - f_ng\|_{L^1} + \|f_ng - fg\|_{L^1}\\ &\leq \|f_n\|_{L^p}\|g_n - g\|_{L^q} + \|g\|_{L^q}\|f_n - f\|_{L^p}\\ &\leq M_1\|g_n - g\|_{L^q} + M_2\|f_n - f\|_{L^p}\end{aligned}$$

which goes to 0 as $n \to \infty$.

(2) Since $g_n \to g$ in L^∞, g_n is bounded by M in L^∞-norm, and when arbitrary $\epsilon > 0$ is chosen there is some $N > 0$ so that $\|g_n - g\|_{L^\infty} < \epsilon$,

$$
\begin{aligned}
\|f_n g_n - fg\|_{L^1} &\leq \|f_n g_n - f_n g\|_{L^1} + \|f_n g - fg\|_{L^1} \\
&\leq \|f_n(g_n - g)\|_{L^1} + \|(f_n - f)g\|_{L^1} \\
&\leq \|\epsilon f_n\|_{L^1} + \|M(f_n - f)\|_{L^1} \\
&\leq \epsilon M' + M\|(f_n - f)\|_{L^1}
\end{aligned}
$$

and the result follows from arbitrariness of ϵ.

A-3

Let $x_n' : y \mapsto \langle x_n, y\rangle, H \to \mathbb{C}$ be a sequence of elements in H^*. Since $\sup_n |x_n'(y)| = \sup_n |\langle x_n, y\rangle| \leq M_y < \infty$ for each $y \in H$, by uniform boundedness principle we have $\|x_n'\|$ is bounded by M in n, Therefore, $M \geq \sup_{\{y\in H, \|y\|=1\}} |\langle x_n, y\rangle| \geq |\left\langle x_n, \frac{x_n}{\|x_n\|}\right\rangle| = \|x_n\|$ for every n.

A-4

If $x \in L^P \cap L^r$, then $x\chi_{x>1} \in L^p \subseteq L^q$ and $x\chi_{x\leq 1} \in L^r \subseteq L^q$. By Minkowski's inequlity, $\|x\|_{L^q} \leq \|x\chi_{x>1}\|_{L^q} + \|x\chi_{x\leq 1}\|_{L^q}$, giving us the result.

A-5

Define $d := \inf\{|x - z|, z \in M\}$. Pick a sequence $\{y_n\} \subseteq M$ so that $|x - y_n| < d + \frac{1}{n}$. Apply parallelogram's law to $\frac{1}{2}(x - y_n)$ and $\frac{1}{2}(x - y_m)$, we have $\frac{1}{2}(\|x - y_n\|^2 + \|x - y_m\|^2) = \|x - \frac{y_n+y_m}{2}\|^2 + \frac{1}{4}\|y_n - y_m\|^2$.

Thus, $\frac{1}{4}\|y_n - y_m\|^2 \leq \frac{1}{2}(2d^2 + \frac{2d}{n} + \frac{2d}{m} + \frac{1}{n^2} + \frac{1}{m^2}) - d^2$, meaning $\{y_n\}$ is Cauchy and hence $y_n \to y \in M$. Therefore, $d \leq \|x - y\| \leq \|x - y_n\| + \|y - y_n\| \to d$ as $n \to \infty$, which shows that y is closest to x than any other element in M is.

If both y, y' minimize the distance to x, then by parallelogram's law applied to $x - y, x - y'$, we have $4d^2 = \|2x - y - y'^2\|^2 + \|y - y'\|^2 = 4\|x - \frac{y+y'}{2}^2\|^2 + \|y - y'\|^2 \geq 4d^2 + \|y - y'\|^2$, forcing $y = y'$.

B-6

To solve

$$\begin{aligned}1-\cos(a+ib) &= 1-\frac{1}{2}(e^{i(a+ib)}+e^{i(a-ib)})\\ &= 1-\frac{1}{2}(e^{-b}(\cos(a)+i\sin(a))+e^{b}(\cos(a)+i\sin(a)))=0,\end{aligned}$$

first we observe that the imaginary part of both sides are zero, so $\sin(a) = 0$. If $\cos(a) = -1$, then we have $1 + \frac{1}{2}(e^{-b} + e^{b}) = 0$, which is impossible. If $\cos(a) = 1$, then we have $2 = e^{b} + e^{-b}$, so $b = 0$. Therefore the zeros for $1 - \cos(z)$ are $\{2k\pi : k \in \mathbb{Z}\}$.

B-7

(i) The only isolated singularity is -1. Let $z = -1 + bi$, we have

$$\begin{aligned}|\sin(\frac{z}{z+1})| &= |\sin(\frac{-1+bi}{bi})| = |\sin 1+\frac{i}{b}|\\ &= |\frac{1}{2i}(e^{i(1+\frac{i}{b})}-e^{-i(1+\frac{i}{b})})|\\ &= |\frac{1}{2i}(e^{i+\frac{-1}{b}}-e^{-i+\frac{1}{b}})| \to \infty \qquad \text{as } b\to 0+.\end{aligned}$$

However, if we let $z = -1 + a$,

$$|\sin(\frac{z}{z+1})| = |\sin(1-1/a)| \le 1 \text{ for every } a \neq 1.$$

Thus 0 is neither a pole or a removable singularity $\Rightarrow$ it must be an essential singularity.

(ii)$\sin(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(z-1)^n = \sum_{n=0}^{\infty}(-1)^n\frac{f^{(n)}(1)}{n!}(1-z)^n$ for every $z \in \mathbb{C}$. Let $z = \frac{w}{w+1} = 1 - \frac{1}{1+w}$, we have

$$\sin(\frac{w}{w+1}) = \sum_{n=0}^{\infty}(-1)^n\frac{f^{(n)}(1)}{n!}(w+1)^{-n} = \sum_{n=0}^{-\infty}(-1)^n\frac{f^{(-n)}(1)}{(-n)!}(w+1)^n.$$

(iii) The coefficient of $\frac{1}{1+w}$ is $-f'(1) = -\cos(\frac{1}{2})\cdot\frac{z+1-z}{(1+z)^2}|_{z=1} = -\frac{1}{4}\cos(\frac{1}{2})$.

B-8

See B-10 of Spring 2011.

B-9

See B-6 of Fall 2007.

B-10

Rouche's theorem states that if f and g are holomorphic on a closed disc D with circle C as its boundary, and $|f(z)| > |g(z)|$ on C, then f and $f+g$ has the same number of zeros inside C.

Take $f(z) = 8$, $g(z) = 2z^5 - z^3 + 3z^2 - z$, we find that $|f| > |g|$ on $\{z : |z| = 1\}$, thus $f + g = 2z^5 - z^3 + 3z^2 - z + 8$ has the same zeros as f on $\{z : |z| < 1\}$, namely no zeros. Since $|2z^5 - z^3 + 3z^2 - z| \neq 8$, $f(z) + g(z) \neq 0$ when $|z| = 1$. By the fundamental theorem of algebra, $f(z) + g(z)$ has 5 zeros on $\mathbb{C}$, thus it also has 5 zeros on $\{z : |z| > 1\}$.