A-1

Note that \( \{ x : f(x) > a \} = \bigcap_{b \in \mathbb{Q}, b < a} \{ x : f(x) > b \} \) is measurable, \( \{ x : f(x) < a \} = \bigcap_{b \in \mathbb{Q}, b > a} \{ x : f(x) < b \} \) is measurable, and \( \{ x : f(x) \in O \} \) is measurable for every open set \( O \) in \( \mathbb{R} \) since \( O \) may be decomposed into a countable union of open intervals.

Also, \( F := \{ A : f^{-1}(A) \text{ is measurable} \} \) is a sigma-algebra containing all open sets in \( \mathbb{R} \). It means that \( \mathcal{B}(\mathbb{R}) \subseteq F \). Thus, for every \( B \in \mathcal{B}(\mathbb{R}) \), \( f^{-1}(B) \) is measurable and we’re done.

A-2

(1)

\[
\| f_n g_n - f g \|_{L^1} \leq \| f_n g_n - f_n g \|_{L^1} + \| f_n g - f g \|_{L^1} \\
\leq \| f_n \|_{L^p} \| g_n - g \|_{L^q} + \| g \|_{L^q} \| f_n - f \|_{L^p} \\
\leq M_1 \| g_n - g \|_{L^q} + M_2 \| f_n - f \|_{L^p}
\]

which goes to 0 as \( n \to \infty \).

(2) Since \( g_n \to g \) in \( L^\infty \), \( g_n \) is bounded by \( M \) in \( L^\infty \)-norm, and when arbitrary \( \epsilon > 0 \) is chosen there is some \( N > 0 \) so that \( \| g_n - g \|_{L^\infty} < \epsilon \),
\[ \|f_n g_n - f g\|_{L^1} \leq \|f_n g_n - f_n g\|_{L^1} + \|f_n g - f g\|_{L^1} \]
\[ \leq \|f_n (g_n - g)\|_{L^1} + \|(f_n - f)g\|_{L^1} \]
\[ \leq \epsilon f_n \|_{L^1} + M (f_n - f) \|_{L^1} \]
\[ \leq \epsilon M' + M \|f_n - f\|_{L^1} \]

and the result follows from arbitrariness of \( \epsilon \).

**A-3**

Let \( x_n' : y \mapsto \langle x_n, y \rangle, H \to \mathbb{C} \) be a sequence of elements in \( H^* \). Since \( \sup_n |x_n'(y)| = \sup_n |\langle x_n, y \rangle| \leq M_y < \infty \) for each \( y \in H \), by uniform boundedness principle we have \( \|x_n'\| \) is bounded by \( M \) in \( n \), Therefore, \( \|x_n'\| \geq \sup_{\{y \in H, \|y\| = 1\}} |\langle x_n, y \rangle| = \|x_n\| \) for every \( n \).

**A-4**

If \( x \in L^p \cap L^r \), then \( x \chi_{x>1} \in L^p \subseteq L^q \) and \( x \chi_{x\leq 1} \in L^r \subseteq L^q \). By Minkowski’s inequality, \( \|x\|_{L^q} \leq \|x \chi_{x>1}\|_{L^q} + \|x \chi_{x\leq 1}\|_{L^q} \), giving us the result.

**A-5**

Define \( d := \inf\{|x - z|, z \in M\} \). Pick a sequence \( \{y_n\} \subseteq M \) so that \( |x - y_n| < d + \frac{1}{n} \). Apply parallelogram’s law to \( \frac{1}{2}(x - y_n) \) and \( \frac{1}{2}(x - y_m) \), we have \( \frac{1}{2}(\|x - y_n\|^2 + \|x - y_m\|^2) = \|x - \frac{y_n + y_m}{2}\|^2 + \frac{1}{4}\|y_n - y_m\|^2 \).

Thus, \( \frac{1}{4}\|y_n - y_m\|^2 \leq \frac{1}{2}(2d^2 + \frac{2d}{n} + \frac{2d}{m} + \frac{1}{n^2} + \frac{1}{m^2}) - d^2 \), meaning \( \{y_n\} \) is Cauchy and hence \( y_n \to y \in M \). Therefore, \( d \leq \|x - y\| \leq \|x - y_n\| + \|y - y_n\| \to d \) as \( n \to \infty \), which shows that \( y \) is closest to \( x \) than any other element in \( M \) is.

If both \( y, y' \) minimize the distance to \( x \), then by parallelogram’s law applied to \( x - y, x - y' \), we have \( 4d^2 = \|2x - y - y'\|^2 + \|y - y'\|^2 = 4\|x - \frac{y + y'}{2}\|^2 + \|y - y'\|^2 \geq 4d^2 + \|y - y'\|^2 \), forcing \( y = y' \).
To solve
\[ 1 - \cos(a + ib) = 1 - \frac{1}{2}(e^{i(a+ib)} + e^{i(a-ib)}) = 1 - \frac{1}{2}(e^{-b}(\cos(a) + i \sin(a)) + e^{b}(\cos(a) + i \sin(a))) = 0, \]

first we observe that the imaginary part of both sides are zero, so \( \sin(a) = 0 \). If \( \cos(a) = -1 \), then we have \( 1 + \frac{1}{2}(e^{-b} + e^{b}) = 0 \), which is impossible. If \( \cos(a) = 1 \), then we have \( 2 = e^{b} + e^{-b} \), so \( b = 0 \). Therefore the zeros for \( 1 - \cos(z) \) are \( \{2k\pi : k \in \mathbb{Z}\} \).

(i) The only isolated singularity is \( -1 \). Let \( z = -1 + bi \), we have
\[
|\sin\left(\frac{z}{z+1}\right)| = |\sin\left(\frac{-1+bi}{bi}\right)| = |\sin 1 + \frac{i}{b}|
\]
\[
= \left|\frac{1}{2i}(e^{i(1+\frac{i}{2})} - e^{-i(1+\frac{i}{2})})\right|
\]
\[
= \left|\frac{1}{2i}(e^{i-\frac{1}{2}} - e^{-i+\frac{1}{2}})\right| \to \infty \quad \text{as } b \to 0 + .
\]
However, if we let \( z = -1 + a \),
\[
|\sin\left(\frac{z}{z+1}\right)| = |\sin(1 - 1/a)| \leq 1 \quad \text{for every } a \neq 1.
\]
Thus \( 0 \) is neither a pole or a removable singularity \( \Rightarrow \) it must be an essential singularity.

(ii) \( \sin(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z - 1)^n = \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n)}(1)}{n!}(1 - z)^n \) for every \( z \in \mathbb{C} \). Let \( z = \frac{w}{w+1} = 1 - \frac{1}{1+w} \), we have
\[
\sin\left(\frac{w}{w+1}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n)}(1)}{n!}(w + 1)^{-n} = \sum_{n=0}^{\infty} (-1)^n \frac{f^{(-n)}(1)}{(-n)!}(w + 1)^{n}.
\]
(iii) The coefficient of \( \frac{1}{1+w} \) is \( -f'(1) = -\cos(\frac{1}{2}) \cdot \frac{z+1-z}{(1+z)^2} \mid_{z=1} = -\frac{1}{4} \cos(\frac{1}{2}). \)
B-8

See B-10 of Spring 2011.

B-9

See B-6 of Fall 2007.

B-10

Rouche’s theorem states that if $f$ and $g$ are holomorphic on a closed disc $D$ with circle $C$ as its boundary, and $|f(z)| > |g(z)|$ on $C$, then $f$ and $f + g$ has the same number of zeros inside $C$.

Take $f(z) = 8$, $g(z) = 2z^5 - z^3 + 3z^2 - z$, we find that $|f| > |g|$ on $\{ z : |z| = 1 \}$, thus $f + g = 2z^5 - z^3 + 3z^2 - z + 8$ has the same zeros as $f$ on $\{ z : |z| < 1 \}$, namely no zeros. Since $|2z^5 - z^3 + 3z^2 - z| \neq 8$, $f(z) + g(z) \neq 0$ when $|z| = 1$. By the fundamental theorem of algebra, $f(z) + g(z)$ has 5 zeros on $\mathbb{C}$, thus it also has 5 zeros on $\{ z : |z| > 1 \}$. 

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