A-1

(a) It’s obvious that \( T_a \) is linear. \((T_a(x))_n \leq \sum_{i=1}^{n} |a_i x_i| \leq \sup_n |x_n| \cdot \|a\|_1.\) Therefore, \( \sup_n |(T_a(x))_n| \leq \sup_n |x_n| \cdot \|a\|_1 \), so \( T_a \) is bounded.

(b) We have shown \( \|T_a\| \leq \|a\|_1 \) in (a). Choose \( y \in l^\infty \) s.t. \( y_i = \text{sgn}(a_i) \), we have \( \|T_a\| \geq \|T_a(y)\|_\infty = \sup_n \sum_{i=1}^{n} |a_i| = \|a\|_1 \).

A-2

(a) By Holder’s inequality, \( \|f\|_{L^2(X)} \|1\|_{L^2(X)} \geq \|f\|_{L^1(X)} \). for every \( f \in L^2(X) \subset L^1(X) \). Thus, \( \|i\| \leq \|1\|_{L^2(X)} = \mu(X)^{1/2} \). On the other hand, \( \|i\| \geq \|i(\mu(X)^{-1/2})\|_{L^1(X)} = \mu(X)^{1/2} \).

(b) \( L^1(X) = L^2(X) \) implies that \( L^1(X) \subset L^2(X) \). See the solution of A-2 in Spring 2010 qualifying exam.

A-3

(\(\Rightarrow\))Let \( f = cg \). For every \( h \in H \) s.t \((g, h) = 0\), we have \((f, h) = c(g, h) = 0\), so there exists no \( h \) that satisfies both conditions.

(\(\Leftarrow\)) Define \( G = \text{span}\{g\} \), which is closed linear subspace space in \( H \), we may write \( H = G \oplus G^\perp \) and \( G^\perp \) is closed. Let \( f \neq cg \) for every \( c \in \mathbb{C} \), we may write \( f = g_0 + g_1 \), where \( g_0 \in G \) and \( g_1 \in G^\perp \), \( g_1 \neq 0 \). Define \( h := \frac{g_1}{\|g_1\|} \) and we may find that \( h \) satisfies \((g, h) = 0\) for all \( g \in G \) and \((f, h) = 1\).
A-4

Note that on \((x,y) \in [0,1] \times [0,1]\), \(|f(x)\chi_{x>y}| \leq |f(x)|\), and \(\infty > \int_0^1 \int_0^1 |f(x)| \, dx \, dy = \int_{[0,1] \times [0,1]} |f(x)| \, d(x \times y)\) by Tonelli’s theorem, which shows \(f(x) \in L^1([0,1] \times [0,1])\) and so is \(f(x)\chi_{x>y}\). Therefore, by Fubini’s theorem, \(\int_{[0,1]} x f(x) \, dx = \int_{[0,1]} \int_{[0,1]} f(x)\chi_{x>y} \, dy \, dx = \int_{[0,1]} \int_{[y,1]} f(x) \, dx \, dy\).

A-5

Assume that \(\Phi(f) = k \neq 0\) for some \(f \in C_0(\mathbb{R})\) (Since \(C_0(\mathbb{R})\) is dense in \(C_0(\mathbb{R})\) in \sup\-norm.) Assume that \(\text{supp}(f) \subset (-M,M)\). Consider \(f_n := \sum_{j=-n}^{n} f(x + 2jM)\), and it is easy to find that \(\|f\|_{C_0(\mathbb{R})} = \|f_n\|_{C_0(\mathbb{R})}\) for all \(n\). However, \(\Phi(f_n) = (2n + 1)k\), and this shows \(\Phi\) is not bounded.

B-6

Let \(C_R = \text{Re}^{i\theta}, \theta\) goes from 0 to \(\pi\). Since \(f(z) = \frac{e^{i2z} - 1 - iz}{\pi^2}\) is holomorphic on \(\mathbb{C}\) (except for \(z = 0\), but it is a removable singularity so we can redefine \(f\)). Therefore, \(\int_{-R}^{R} \frac{\cos(2x)-1}{x^2} \, dx + i \int_{-R}^{R} \frac{\sin(2x)-2x}{x^2} \, dx + \int_{C_R} \frac{e^{i2z} - 1 - iz}{z^2} \, dz = 0\) \((*)\).

Since \(|\int_{C_R} \frac{e^{i2z}}{z^2} \, dz| \leq \pi R \cdot \frac{2}{R^2}\), and \(\int_{C_R} \frac{-iz}{z^2} \, dz = -2i \int_{C_R} \frac{1}{z} \, dz = 2\pi\), we let \(R \to \infty\) and take real parts in \((*)\) to obtain

\[
\int_{-\infty}^{\infty} \frac{\cos(2x)-1}{x^2} \, dx + 2\pi = 0 = -2 \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} \, dx + 2\pi.
\]

It follows that \(\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} \, dx = \pi \Rightarrow \int_{0}^{\infty} \frac{\sin^2(x)}{x^2} \, dx = \pi/2\).

B-7

Assume not. Let \(D \setminus \{a\}\) be a deleted neighborhood of \(a\). Then there is some \(r > 0, w \in \mathbb{C}\), so that \(|f(z) - w| > r\) for all \(z \in D \setminus \{a\}\). Define \(g(z) = \frac{1}{f(z)-w}\), we have that \(|g(z)| < \frac{1}{r}\) for \(z \in D \setminus \{a\}\). As a result, \(a\) is a removable singularity of \(g\). If \(g(a) = 0\), then \(a\) is a pole for \(f\), a contradiction. If \(g(a) \neq 0\), then \(a\) is a removable singularity for \(f\), which is again a contradiction.
Approach 1 Apply Rouche’s theorem on the circle \(|z - R|^2 = R^2\) for arbitrary \(R > 0\).

Approach 2 Let \(z = a + bi\). We may rewrite the equation as \(\alpha - a - e^{-a}\cos(b) - ib + i\sin(b) = 0\). If \(\pi/2 > b > 0\), then by MVT, \(|\sin(b)/b| = |\cos(b')| < 1\), where \(b'\) lies between 0 and \(b\). If \(b \geq \pi/2\), then we also have \(|\sin(b)/b| < 1\). Similar for \(b < 0\). As a result, 0 is the only real root for \(\sin(b) = b\), and the equation becomes

\[
\alpha - a - e^{-a} = 0.
\]

Let \(f(x) = x + e^{-x}\). Since \(f(0) = 1 < \alpha\) and \(f(\alpha) > \alpha\), by intermediate value theorem we can find \(x'\) so that \(f(x') = \alpha\), which proves the existence of this equation. To prove uniqueness, we consider \(x' > y' > 0\) be two different solutions, and we have

\[
x' - y' = e^{-y'} - e^{-x'} = e^{-z'}(x' - y') < x' - y'
\]

for some \(y' < z' < x'\), by MVT, and this is a contradiction. This proves the uniqueness.

Since \(\phi\) is a bijective (and also holomorphic) map, the inverse \(\phi^{-1}\) exists, and it is continuous by open mapping theorem. Therefore, \(w = \phi(z) \to w' = \phi(z')\) implies \(z \to z'\), and we have

\[
\lim_{w \to w'} \frac{\phi^{-1}(w) - \phi^{-1}(w')}{w - w'} = \lim_{w \to w'} \frac{w - w'}{\phi^{-1}(w) - \phi^{-1}(w')}^{-1} = \lim_{w \to w'} \frac{\phi(z) - \phi(z')}{z - z'}^{-1} = (\phi'(z'))^{-1} = (\phi'(\phi^{-1}(w')))^{-1},
\]

which is well-defined because \(\phi'(z) \neq 0\) for all \(z \in \Omega\).

Let \(f : \Omega \to D\) be an arbitrary holomorphic map. \(f \circ \phi^{-1}\) is a holomorphic map from \(D \to D\), so by Schwarz’s lemma we have \(1 \geq |(f \circ \phi^{-1})'(0)|\).
However, there are only finitely such $z$’s, for otherwise $\{\frac{1}{z} : g(z) = 0\}$ would have a cluster point in the unit disk, and $f \equiv 0$ by uniqueness theorem. Let these $z$’s be $\{z_1, \cdots, z_n\}$, and they are poles of $h$.

Define $F(z) = f(z)$ when $|z| \leq 1$, and $F(z) = h(z)$ when $|z| > 1, z \notin \{z_1, \cdots, z_n\}$. Since $F(z)$ is holomorphic on $\mathbb{C} \setminus \{z : |z| = 1\}$.
and $F(z)$ is continuous on $|z| = 1$ by the fact that $|f(z)| = 1$ on $|z| = 1$, Morera’s theorem shows us $F(z)$ is holomorphic on $\mathbb{C} \setminus \{z_1, \cdots, z_n\}$, and the proof is complete.