Real & Complex Analysis Qualifying Exam Solution, Fall 2008

Shiu-Tang Li

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A-1

(a) It's obvious that T_a is linear. $(T_a(x))_n \leq \sum_{i=1}^n |a_i x_i| \leq \sup_n |x_n| \cdot \|a\|_1$. Therefore, $\sup_n |(T_a(x))_n| \leq \sup_n |x_n| \cdot \|a\|_1$, so T_a is bounded.

(b) We have shown $\|T_a\| \leq \|a\|_1$ in (a). Choose $y \in l^\infty$ s.t. $y_i = sgn(a_i)$, we have $\|T_a\| \geq \|T_a(y)\|_\infty = \sup_n \sum_{i=1}^n |a_i| = \|a\|_1$.

A-2

(a) By Holder's inequality, $\|f\|_{L^2(X)}\|1\|_{L^2(X)} \geq \|f\|_{L^1(X)}$. for every $f \in L^2(X) \subset L^1(X)$. Thus, $\|i\| \leq \|1\|_{L^2(X)} = \mu(X)^{1/2}$. On the other hand, $\|i\| \geq \|i(\mu(X)^{-1/2})\|_{L^1(X)} = \mu(X)^{1/2}$.

(b) $L^1(X) = L^2(X)$ implies that $L^1(X) \subset L^2(X)$. See the solution of A-2 in Spring 2010 qualifying exam.

A-3

($\Rightarrow$)Let $f = cg$. For every $h \in H$ s.t $(g, h) = 0$, we have $(f, h) = c(g, h) = 0$, so there exists no h that satisfies both conditions.

($\Leftarrow$) Define $G = span\{g\}$, which is closed linear subspace space in H, we may write $H = G \oplus G^\perp$ and $G^\perp$ is closed. Let $f \neq cg$ for every $c \in \mathbb{C}$, we may write $f = g_0 + g_1$, where $g_0 \in G$ and $g_1 \in G^\perp, g_1 \neq 0$. Define $h := \frac{g_1}{\|g_1\|^2}$ and we may find that h satisfies $(g, h) = 0$ for all $g \in G$ and $(f, h) = 1$.

A-4

Note that on $(x, y) \in [0,1] \times [0,1]$, $|f(x)\chi_{x>y}| \leq |f(x)|$, and $\infty > \int_0^1 \int_0^1 |f(x)|\, dx\, dy = \int_{[0,1]\times[0,1]} |f(x)|\, d(x \times y)$ by Tonelli's theorem, which shows $f(x) \in L^1([0,1]\times[0,1])$ and so is $f(x)\chi_{x>y}$. Therefore, by Fubini's theorem, $\int_{[0,1]} xf(x)\, dx = \int_{[0,1]} \int_{[0,1]} f(x)\chi_{x>y}\, dy\, dx = \int_{[0,1]} \int_{[0,1]} f(x)\chi_{x>y}\, dx\, dy = \int_{[0,1]} \int_{[y,1]} f(x)\, dx\, dy$.

A-5

Assume that $\Phi(f) = k \neq 0$ for some $f \in C_c(\mathbb{R})$ (Since $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ in sup-norm.) Assume that $supp(f) \subset (-M, M)$. Consider $f_n := \sum_{j=-n}^{n} f(x + 2jM)$, and it is easy to find that $\|f\|_{C_0(\mathbb{R})} = \|f_n\|_{C_0(\mathbb{R})}$ for all n. However, $\Phi(f_n) = (2n+1)k$, and this shows Φ is not bounded.

B-6

Let $C_R = Re^{i\theta}$, θ goes from 0 to π. Since $f(z) = \frac{e^{i2z}-1-i2z}{z^2}$ is holomorphic on $\mathbb{C}$ (except for $z = 0$, but it is a removable singularity so we can redefine f). Therefore, $\int_{-R}^{R} \frac{\cos(2x)-1}{x^2}\, dx + i \int_{-R}^{R} \frac{\sin(2x)-2x}{x^2}\, dx + \int_{C_R} \frac{e^{i2z}-1-i2z}{z^2}\, dz = 0$...(*).

Since $|\int_{C_R} \frac{e^{i2z}-1}{z^2}\, dz| \leq \pi R \cdot \frac{2}{R^2}$, and $\int_{C_R} \frac{-i2z}{z^2}\, dz = -2i \int_{C_R} \frac{1}{z}\, dz = 2\pi$, we let $R \to \infty$ and take real parts in (*) to obtain

$$\int_{-\infty}^{\infty} \frac{\cos(2x) - 1}{x^2}\, dx + 2\pi = 0 = -2 \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\, dx + 2\pi.$$

It follows that $\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\, dx = \pi \Rightarrow \int_0^{\infty} \frac{\sin^2(x)}{x^2}\, dx = \pi/2$.

B-7

Assume not. Let $D \setminus \{a\}$ be a deleted neighborhood of a. Then there is some $r > 0$, $w \in \mathbb{C}$, so that $|f(z) - w| > r$ for all $z \in D \setminus \{a\}$. Define $g(z) = \frac{1}{f(z)-w}$, we have that $|g(z)| < \frac{1}{r}$ for $z \in D \setminus \{a\}$. As a result, a is a removable singularity of g. If $g(a) = 0$, then a is a pole for f, a contradiction. If $g(a) \neq 0$, then a is a removable singularity for f, which is again a contradiction.

B-8

Approach 1 Apply Rouche's theorem on the circle $|z - R|^2 = R^2$ for arbitrary $R > 0$.

Approach 2 Let $z = a + bi$. We may rewrite the equation as $\alpha - a - e^{-a}\cos(b) - ib + i\sin(b) = 0$. If $\pi/2 > b > 0$, then by MVT, $|\sin(b)/b| = |\cos(b')| < 1$, where b' lies between 0 and b. If $b \geq \pi/2$, then we also have $|\sin(b)/b| < 1$. Similar for $b < 0$. As a result, 0 is the only real root for $\sin(b) = b$, and the equation becomes

$$\alpha - a - e^{-a} = 0.$$

Let $f(x) = x + e^{-x}$. Since $f(0) = 1 < \alpha$ and $f(\alpha) > \alpha$, by intermediate value theorem we can find x' so that $f(x') = \alpha$, which proves the existence of this equation. To prove uniqueness, we consider $x' > y' > 0$ be two different solutions, and we have

$$x' - y' = e^{-y'} - e^{-x'} = e^{-z'}(x' - y') < x' - y'$$

for some $y' < z' < x'$, by MVT, and this is a contradiction. This proves the uniqueness.

B-9

Since ϕ is a bijective (and also holomorphic) map, the inverse ϕ^{-1} exists, and it is continuous by open mapping theorem. Therefore, $w = \phi(z) \to w' = \phi(z')$ implies $z \to z'$, and we have

$$\begin{aligned}
\lim_{w \to w'} \frac{\phi^{-1}(w) - \phi^{-1}(w')}{w - w'} &= \lim_{w \to w'} \left(\frac{w - w'}{\phi^{-1}(w) - \phi^{-1}(w')} \right)^{-1} \\
&= \lim_{w \to w'} \left(\frac{\phi(z) - \phi(z')}{z - z'} \right)^{-1} \\
&= (\phi'(z'))^{-1} = \Big(\phi'(\phi^{-1}(w')) \Big)^{-1},
\end{aligned}$$

which is well-defined because $\phi'(z) \neq 0$ for all $z \in \Omega$.

Let $f : \Omega \to D$ be an arbitrary holomorphic map. $f \circ \phi^{-1}$ is a holomorphic map from $D \to D$, so by Schwarz's lemma we have $1 \geq |(f \circ$

$\phi^{-1})'(0)| = |f'(\phi^{-1}(0))(\phi^{-1})'(0)| = |f'(\phi^{-1}(0))\Big(\phi'(\phi^{-1}(0))\Big)^{-1}|$. Therefore, $|\phi'(z_0)| \geq |f'(\phi^{-1}(0))| = |f'(z_0)|$.

B-10

We would prove that $g(z) = \overline{f(1/\overline{z})} = U(z) + iV(z)$ is holomorphic on $|z| > 1$. To see this, we let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, we have $g(z) = u(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) - iv(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$, and for $|z| > 1$,

$$\begin{aligned}
\frac{\partial}{\partial x}\Big(u(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})\Big) &= u_x(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) \cdot \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} \\
&\quad + u_y(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) \cdot \frac{-2xy}{(x^2+y^2)^2} \\
&= v_y(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) \cdot \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + v_x(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) \cdot \frac{2xy}{(x^2+y^2)^2} \\
&= \frac{\partial}{\partial y}\Big(-v(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})\Big),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial y}\Big(u(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})\Big) &= u_x(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) \cdot \frac{-2xy}{(x^2+y^2)^2} \\
&\quad + u_y(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) \cdot \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \\
&= v_y(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) \cdot \frac{-2xy}{(x^2+y^2)^2} - v_x(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) \cdot \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \\
&= \frac{\partial}{\partial x}\Big(v(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})\Big).
\end{aligned}$$

Therefore, we have $U_x(z), U_y(z), V_x(z), V_y(z)$ all exist and are continuous on $|z| > 1$, and for each $|z| > 1$, $U_x(z) = V_y(z)$ and $U_y(z) = -V_x(z)$. This proves $g(z) = \overline{f(1/\overline{z})}$ is holomorphic on $|z| > 1$, and hence $h(z) := \Big(\overline{f(1/\overline{z})}\Big)^{-1}$ is holomorphic on $|z| > 1$ except for those z such that $g(z) = \overline{f(1/\overline{z})} = 0$. However, there are only finitely such z's, for otherwise $\{\frac{1}{\overline{z}} : g(z) = 0\}$ would have a cluster point in the unit disk, and $f \equiv 0$ by uniqueness theorem. Let these z's be $\{z_1, \cdots, z_n\}$, and they are poles of h.

Define $F(z) = f(z)$ when $|z| \leq 1$, and $F(z) = h(z)$ when $|z| > 1, z \notin \{z_1, \cdots, z_n\}$. Since $F(z)$ is holomorphic on $\mathbb{C} \setminus \{z : |z| = 1\} \cup \{z_1, \cdots, z_n\}$,

and $F(z)$ is continuous on $|z| = 1$ by the fact that $|f(z)| = 1$ on $|z| = 1$, Morera's theorem shows us $F(z)$ is holomorphic on $\mathbb{C} \setminus \{z_1, \cdots, z_n\}$, and the proof is complete.