A-1

Apply dominated convergence theorem with dominating function $|f|$ on every $h_n \to 0$ to prove $F(x + h_n) \to F(x)$. Arbitrariness of $\{h_n\}$ implies that $F$ is continuous on $x$ for every $x \in \mathbb{R}$.

A-2

We have $\|Mg\|_2 = (\int_{\mathbb{R}} f^2 g^2 \, dx)^{1/2} \leq \|f\|_\infty (\int_{\mathbb{R}} g^2 \, dx)^{1/2} = \|f\|_\infty \|g\|_{L^2(\mathbb{R})}$.

To see $\|M\| \geq \|f\|_\infty$, consider $g = \text{sgn}(f) \chi_{[-n,n]} \chi_{\{|f| > \|f\|_\infty - \epsilon\}}$, where $n = n_\epsilon$ is chosen s.t. $\mu(\{|f| > \|f\|_\infty - \epsilon, x \in [-n,n]\}) > 0$.

Therefore, $\|Mg\|_2^2 = \int_{-n}^n |f|^2 \chi_{\{|f| > \|f\|_\infty - \epsilon\}} \, d\mu \geq (\|f\|_\infty - \epsilon)^2 \mu(\{|f| > \|f\|_\infty - \epsilon, x \in [-n,n]\}) = \|g\|_{L^2(\|f\|_\infty - \epsilon)^2}$. Since $\epsilon$ is arbitrary, we have $\|M\| \geq \|f\|_\infty$.

A-3

(a) Let $Tv = \lambda v, v \neq 0$. Since $(Tv, v) = \lambda (v, v) = (v, Tv) = \overline{\lambda} (v, v)$ and $(v, v) > 0$, $\lambda$ is real.

(b) Let $Tu = \mu u, Tv = \lambda v, \mu \neq \lambda$ and $u, v \neq 0$. We have $(Tu, v) = \mu (u, v) = (u, Tv) = \lambda (u, v)$. $\mu \neq \lambda \Rightarrow (u, v) = 0$.

(c) For each $\lambda_n$, we pick some $\|x_n\| = 1$ so that $Tx_n = \lambda_n x_n$. We claim that $X = \{\sum_{n=1}^{\infty} a_n x_n : \sum_{n=1}^{\infty} |a_n|^2 < \infty\}$ is a closed subspace of $H$. For if
\[ y_k := \sum_{n=1}^{\infty} a_{kn} x_n \text{ is Cauchy, } a_{kn} \to b_n \text{ as } k \to \infty \text{ for each } n \in \mathbb{N}. \text{ Given } \epsilon > 0, \text{ we have } \|y_m - y_n\| < \epsilon \text{ for all } m, n > K, \text{ and now we apply Fatou’s lemma to get}
\]
\[ \sum_{n=1}^{\infty} |b_n - a_{Kn}|^2 \leq \liminf_{k \to \infty} \sum_{n=1}^{\infty} |a_{kn} - a_{Kn}|^2 \]
\[ = \liminf_{k \to \infty} \|y_k - y_K\| \leq \epsilon. \]

Besides, we may pick \( K \) large so that \( \left( \sum_{n=1}^{\infty} |b_n|^2 \right)^{1/2} \leq \left( \sum_{n=1}^{\infty} |b_n - a_{Kn}|^2 \right)^{1/2} \) \( + \left( \sum_{n=1}^{\infty} |a_{Kn}|^2 \right)^{1/2} < \infty \). That is, \( \sum_{n=1}^{\infty} b_n x_n \in X \). That \( y_k \to \sum_{n=1}^{\infty} b_n x_n \) in \( X \) is equivalent to the fact that \( X \) is closed.

Since \( X \) is closed, we may decompose \( H = X \oplus X^\perp \). We claim that \( x_n \to 0 \). First, for each \( y \in H \) we may write it as \( \sum_{m=1}^{\infty} a_m x_m + x' \), where \( \sum_{m=1}^{\infty} a_m x_m \in X \) and \( x' \in X^\perp \). We have
\[ (x_n, y) = (x_n, \sum_{m=1}^{\infty} a_m x_m + x') = (x_n, \sum_{m=1}^{\infty} a_m x_m) \]
\[ = (x_n, \sum_{m=1}^{n} a_m x_m) + (x_n, \sum_{m=n+1}^{\infty} a_m x_m) \]
\[ = a_n + (x_n, \sum_{m=n+1}^{\infty} a_m x_m) \to 0 \text{ as } n \to \infty. \]

The above convergence is due to \( \sum_{m=1}^{\infty} |a_m|^2 < \infty \). We have shown our claim.

Next, since \( \{x_n\}_n \) is a bounded sequence, \( \{T x_n\}_n \) is precompact in \( H \). We claim that \( T x_n \to 0 \). If not, by precompactness of \( \{T x_n\}_n \) we can find a subsequence \( \{T x_{m_k}\}_k \) so that \( T x_{m_k} \to z \neq 0 \Rightarrow T x_{m_k} \to z \neq 0 \). However, \( x_n \to 0 \) implies \( T x_n \to 0 \) by that \( T \) is adjoint, which is a contradiction. Therefore,
\[ T x_n \to 0 \Rightarrow \|T x_n\| \to 0 \]
\[ \Rightarrow \|\lambda_n x_n\| \to 0 \]
\[ \Rightarrow \|\lambda_n\| \to 0 \]
\[ \Rightarrow \lambda_n \to 0. \]

And the proof is complete.
A-4

See the sol A−4 in spring 2008 real and complex analysis qualifying exam. (Or directly from Rudin’s book.)

A-5

\[ \|Tg\|_2 = \left( \int_0^1 (\int_0^1 (Tg(x))^2 \, dx)^{1/2} \right)^2 \leq \left( \int_0^1 (\int_0^1 K(x,y)^2 \, dy) \right)^{1/2} \left( \int_0^1 (\int_0^1 f(y)^2 \, dy) \, dx \right)^{1/2} \]

\[ = \|f\|_{L^2(I)} \int_I \int_I K(x,y)^2 \, dy \, dx \]

\[ = \|f\|_{L^2(I)} \int_{I \times I} K(x,y)^2 \, d(x \times y) \]

where the last identity is due to Tonelli’s theorem.

B-6

I would prove further that if \( f(z) \leq A + B|z|^n \) for some \( A \geq 0, B > 0 \) and for all \( z \in \mathbb{C} \), then \( f(z) \) is a polynomial of degree \( \leq n \).

we first note that \( g(z) := \frac{f(z)-f(0)}{z} \) for \( z \neq 0 \) and \( g(0) = f'(0) \) is also an entire function, for \( g \) is holomorphic on \( \mathbb{C} \setminus \{0\} \) and is continuous on \( \mathbb{C} \), thus we may prove holomorphicity of \( g \) on \( \mathbb{C} \) using Morera’s theorem.

The proof is by induction. When \( n = 0 \), this is Liouville’s theorem. For \( n = N > 1 \), assume that \( f(z) \leq A + B|z|^N \), we have \( |g(z)| \leq \frac{A + B|z|^N + |f(0)|}{|z|} \leq A' + B|z|^{N-1} \) when \( |z| > 1 \), and \( |g(z)| \leq A'' + B|z|^{N-1} \) when \( |z| \leq 1 \) since \( |g| \) is bounded on bounded domains. Therefore, by induction hypothesis, \( g(z) \) is a polynomial of degree \( \leq N - 1 \). Since \( g(z)z + f(0) = f(z) \) for every \( z \in \mathbb{C} \), \( f \) is a polynomial of degree \( \leq N \), and this completes the proof.

B-7

(a) Let \( \phi(z) = \frac{z}{1+z} \), which is analytic on \( H \), and for \( z = a + bi, b > 0 \), \( |i-z| = |a+(1-b)i| < |a+(1+b)i| = |i+z| \), which shows that \( \phi : H \to D := \{ z : |z| < 1 \} \). To see \( \phi(z) \) is onto and \( 1-1 \), for any
\[ w = re^{i\theta} \in D, \ r < 1, \ w = \frac{i-z}{1+w} = i^{1-r}e^{i\theta} = i^{1-r^2-2i\sin(\theta)} \in H. \]

Thus, we may define the inverse map of \( \phi \) by \( \phi^{-1}(z) = \frac{i-z}{1+z} : D \to H \). These facts show that \( \phi \) is a conformal map from \( H \) to \( D \), which takes \( i \) to 0.

(b) Consider the holomorphic map \( \phi \circ f \circ \phi^{-1} : D \to D \).
\[
| (\phi \circ f \circ \phi^{-1})'(0) | = | \phi'(f(\phi^{-1}(0))) f'(\phi^{-1}(0))(\phi^{-1})'(0) | \\
= | \phi'(i) f'(i) (\phi^{-1})'(0) | \\
= | \phi'(i) f'(i) \frac{1}{\phi'(\phi^{-1}(0))} | \\
= | \phi'(i) f'(i) \frac{1}{\phi'(i)} | = | f'(i) |.
\]

In the above lines we use the fact that \( | \phi'(i) | \neq 0 \), for \( \phi \) is an injective holomorphic map. By Schwarz’s lemma, we have \( 1 \geq | (\phi \circ f \circ \phi^{-1})'(0) | = | f'(i) |. \) If the equality holds, then \( \phi \circ f \circ \phi^{-1}(z) = e^{i\theta}z \), and thus
\[
f(z) = \phi^{-1} \circ e^{i\theta} z \circ \phi(z) \\
= \phi^{-1} \circ e^{i\theta} \frac{i-z}{i+z} \\
= \frac{1 - e^{i\theta} \frac{i-z}{i+z}}{1 + e^{i\theta} \frac{i-z}{i+z}} \\
= \frac{i + z - e^{i\theta}(i-z)}{i + z + e^{i\theta}(i-z)} \\
= \frac{e^{-i\theta/2}(i + z) - e^{i\theta/2}(i-z)}{e^{-i\theta/2}(i + z) + e^{i\theta/2}(i-z)} \\
= \frac{1}{-i} \cdot \frac{-2i \sin(\theta/2) \cdot i + 2 \cos(\theta/2)z}{2 \cos(\theta/2) \cdot i - 2i \sin(\theta/2)z} \\
= \frac{\sin(\theta/2) + \cos(\theta/2)z}{\cos(\theta/2) - \sin(\theta/2)z}.
\]

B-8

Define \( g(z) = f(z) \) when \( 0 \leq \text{Re} \ z \leq 1 \). Define \( g(z) = \bar{g(2-z)} \) when \( 1 < \text{Re} \ z \leq 2 \). We claim that \( g(z) \) is holomorphic on \( R_{1,2} := \{ z : 1 < \text{Re} \ z < 2 \} \). To see this, for any \( z_0 \in R_{1,2} \), there is some \( \epsilon > 0 \) so that \( B_{z_0}(\epsilon) \subset R_{1,2} \). Since \( g(z) \) is holomorphic on \( R_{0,1} := \{ z : 0 < \text{Re} \ z < 1 \} \), we
have \( g(2 - z) = \sum_{n=0}^{\infty} a_n (2 - z - (2 - z_0))^n = \sum_{n=0}^{\infty} a_n (-1)^n (z - z_0)^n \), which proves the existence of power series at every neighborhood of \( z \in R_{1.2} \) and hence our claim.

Since \( g(z) \) defined in this way is continuous on \( \{ z : 0 < Re z < 2 \} \) due to the fact \( f(1 + ix) \in \mathbb{R} \) for ever \( x \in \mathbb{R} \), and \( g(z) \) holomorphic on \( R_{0.1} \cup R_{1.2} \), by Morera’s theorem, \( g(z) \) is holomorphic on \( \{ z : 0 < Re z < 2 \} \). Also, from our definition of \( g \), \( g(ix) = g(2 + ix) \) for every \( x \in \mathbb{R} \), for \( f(ix) \in \mathbb{R} \forall x \in \mathbb{R} \).

Now for every \( 2n \leq Re z \leq 2n + 2, n \in \mathbb{Z} \), we may define \( g(z) = g(z - 2n) \). It is straightforward that \( g \) is holomorphic on \( 2n < Re z < 2n + 2 \). In addition, it is continuous on \( \{ z : Re z = 2n, n \in \mathbb{Z} \} \). By Morera’s theorem, \( g(z) \) is holomorphic on \( \mathbb{C} \).

Since \( f \) and \( g \) coincide on \( R_{0.1} \), \( f \equiv g \) by the uniqueness theorem. As a result, for any \( z \in \mathbb{C} \), \( f(z) = g(z) = g(z + 2) = f(z + 2) \).

**B-9**

For each \( z \in \Omega \), there is an \( \epsilon \)-ball centered at \( z \) and its closure is contained in \( \Omega \). Let \( C \) be its boundary. By Cauchy’s integral formula,

\[
f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta,
\]

\[
\Rightarrow f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta,
\]

for which the proof is omitted. Therefore, for ever \( f \in F, |f'(z)| \leq \frac{1}{2\pi} \cdot \frac{M}{\epsilon^2} \cdot 2\pi \epsilon := B_z \), where \( M \) is chosen that \( |f(w)| < M \) for all \( w \in B_z(\epsilon) \).

**B-10**

Let \( C_R = Re^{i\theta}, \theta \) goes from 0 to \( \pi \). By the residue theorem, \( \int_{-R}^{R} \frac{\cos(x)}{x^2+4} dx + i \int_{-R}^{R} \frac{\sin(x)}{x^2+4} dx + \int_{C_R} \frac{e^{ix}}{x^2+4} dz = 2\pi i Res(f; 2i) = 2\pi i \cdot \frac{e^{-2}}{2i} = \frac{\pi}{2} e^{-2} \). Since \( |\int_{C_R} \frac{e^{ix}}{x^2+4} dz| \leq \pi R \cdot \frac{1}{R^2+4} \), let \( R \to \infty \) we have \( \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+4} dx = \frac{\pi}{2} e^{-2} \).