1

(1)

\[ E[S_N] = E[E[S_N | N]] \]
\[ = E[\sum_{i=0}^{\infty} 1_{\{N=i\}} E[S_i]] \]
\[ = E[\sum_{i=0}^{\infty} 1_{\{N=i\}} i\mu] \]
\[ = \mu E[\sum_{i=0}^{\infty} 1_{\{N=i\}} i] = \mu E[N] \]

(2)

\[ Var[S_N] = E[E[S_N^2 | N]] - (ES_N)^2 \]
\[ = E[\sum_{i=0}^{\infty} 1_{\{N=i\}} E[S_i^2]] - (\mu E[N])^2 \]
\[ = E[\sum_{i=0}^{\infty} 1_{\{N=i\}} (i\sigma^2 + i(i-1)\mu^2)] - \mu^2 (E[N])^2 \]
\[ = \sigma^2 E[N] + \mu^2 (E[N^2] - E[N]) - \mu^2 (E[N])^2 \]
\[ = (\sigma^2 - \mu^2) E[N] + \mu^2 Var[N] \]
(a) Given any \( A > 0 \),

\[
P(|X_n| > A) = 2 \int_A^\infty \frac{n}{\pi(1 + n^2x^2)} dx
= \frac{2}{\pi} \tan^{-1}(nx)|_A^\infty
= \frac{2}{\pi} \left( \frac{\pi}{2} - \tan^{-1}(nA) \right) \rightarrow 0 \text{ as } n \rightarrow \infty
\]

Thus \( X_n \rightarrow 0 \) in pr.
(b) \[ E[|X_n - 0|] = E[|X_n|] = \frac{2}{\pi} \int_0^\infty \frac{nx}{1+n^2x^2} \, dx = \frac{1}{n\pi} \ln(1 + n^2x^2) \bigg|_0^\infty = \infty. \]
Thus \( X_n \not\to 0 \) in \( L^1 \).

(c) Given \( A > 0 \),
\[
\sum_{n=1}^\infty P(|X_n - 0| > A) = \frac{2}{\pi} \sum_{n=1}^\infty \left( \frac{\pi}{2} - \tan^{-1}(nA) \right) 
\geq \frac{2}{\pi} \left( \int_0^\infty \frac{\pi}{2} - \tan^{-1}(x) \, dx - \frac{\pi}{2} \right) 
= \frac{2}{\pi} \left( \int_0^\pi \tan(x) \, dx - \frac{\pi}{2} \right) 
= \frac{2}{\pi} \left( \ln(\sec x) \bigg|_0^\pi - \frac{\pi}{2} \right) = \infty.
\]
By Borel-Cantelli lemma, \( P(|X_n - 0| > A \text{ i.o.}) = 1 \). This implies the failure of a.s. convergence.

4

Hard.

5

(a) \[ E[M_n|F_{n-1}] = \lambda_0^{S_{n-1}} E[\lambda_0^{X_n}|F_{n-1}] = \lambda_0^{S_{n-1}} E[\lambda_0^{X_n}] = \lambda_0^{S_{n-1}}. \]

(b) Since \( E[\lambda_0^{X_1}] = 1 \), we let \( \lambda_0 = a(\cos \theta + i\sin \theta) \) and we have
\[ pa \cos \theta + (1-p)a^{-1} \cos \theta = 1 \quad \text{(1)} \]
and
\[ pa \sin \theta - (1-p)a^{-1} \sin \theta = 0 \quad \text{(2)} \]
by (2), we have either \( a = \sqrt{\frac{1-p}{p}} \) or \( \sin \theta = 0 \). If \( a = \sqrt{\frac{1-p}{p}} \), then by (1) we have \( \cos \theta = \frac{1}{2\sqrt{p(1-p)}} \geq 1 \), and this implies \( \cos \theta = 1 \) and \( p = \frac{1}{2} \Rightarrow a = 1 \).

If we start at the assumption that \( \sin \theta = 0 \), by (1) we have \( \cos \theta = 1 \) to make \( pa \cos \theta + (1-p)a^{-1} \cos \theta \) positive. Hence \( pa^2 - a + (1-p) = 0 \) and we have \( a = 1 \) or \( a = \frac{1-p}{p} \). Therefore, \( \lambda_0 \) equals 1 or \( \frac{1-p}{p} \), and for any \( n \) we have
\[ E[M_n] = E[|X_1|^n] = p|\lambda| + (1-p)|\lambda^{-1}| = p\lambda + (1-p)\lambda^{-1} = 1 \]
which shows \( \{M_n\} \) is \( L^1 \) bounded; martingale convergence theorem tells us the limit exists and is finite a.s.

(c) Let \( X_1 \) takes value on 1 and 3 with probability \( \frac{1}{2} \) each, and \( \lambda_0 = ae^{i\theta} \). We have
\[
\frac{1}{2}a \cos \theta + \frac{1}{2}a^3(4 \cos^3 \theta - 3 \cos \theta) = 1 \tag{3}
\]
and
\[
\frac{1}{2}a \sin \theta + \frac{1}{2}a^3(3 \sin \theta - 4 \sin^3 \theta) = 0 \tag{4}
\]
By (4) we have \( \sin^2 \theta = \frac{3}{4} + \frac{1}{4a^2} \). Therefore we have the equation \( \left(\frac{1}{4a^2}\right)^{\frac{1}{2}} = 1 - \frac{1}{4a^2} \), which solves to \( a = \sqrt{2} \). Then \( \cos \theta = -\frac{1}{\sqrt{2}} \), and we choose \( \sin \theta = \frac{\sqrt{2}}{4} \).

We find that \( E[\lambda_0 X_1] = 1 \) and \( P(|\lambda_0 X_1| = \sqrt{2}) = P(|\lambda_0 X_1| = 2\sqrt{2}) = \frac{1}{2} \). Thus \( |M_n| \geq (\sqrt{2})^n \) a.s., showing that the limit does not exist for a.s. \( \omega \).

6

Given \( i \in \mathbb{N} \), we may choose \( A_i \) s.t. \( P(|X| > A_i) < 1/i \) and \( F_X(x) \) is continuous at both \( A_i \) and \( -A_i \), and \( A_i \to \infty \) as \( i \to \infty \). Therefore,
\[
\int_{\{x>0\}} x \mu = \lim_{i \to \infty} \int_{\{0<x \leq A_i\}} x \mu \quad \text{by monotone convergence theorem}
\]
\[
\leq \lim sup_{i \to \infty} \left| \int_{\{0<x \leq A_i\}} x \mu - \int_{\{0<x \leq A_i\}} x \mu_n \right| + \lim sup_{i \to \infty} \int_{\{0<x \leq A_i\}} x \mu_n \tag{5}
\]
\[
\leq \lim sup_{i \to \infty} \left| \int_{\{0<x \leq A_i\}} x \mu - \int_{\{0<x \leq A_i\}} x \mu_n \right| + (\sup_{n \geq 1} E[X_n^2])^{1/2}
\]
\[
\leq 1 + (\sup_{n \geq 1} E[X_n^2])^{1/2},
\]
if we select \( n = n(i) \) above to satisfy \( \left| \int_{\{0<x \leq A_i\}} x \mu - \int_{\{0<x \leq A_i\}} x \mu_n \right| \leq 1 \) for each \( i \). This can be done because \( X_n \Rightarrow X \) implies \( E[f(X_n)] \to E[f(X)] \) for any bounded continuous function. Similarly we can prove the finiteness of \( \int_{\{x \leq 0\}} x \mu \). This proves \( X \in L^1 \). For the rest part,
\[| \int x \, d\mu_n(x) - \int x \, d\mu(x) | \leq | \int_{|x| > A_i} x \, d\mu_n(x) | + | \int_{|x| > A_i} x \, d\mu(x) |
+ | \int_{|x| \leq A_i} x \, d\mu_n(x) - \int_{|x| \leq A_i} x \, d\mu(x) |
\leq \left( \frac{1}{i} \cdot \sup_{n \geq 1} E[X_n^2] \right)^{1/2} + E[X; |X| > A_i]
+ | \int_{|x| \leq A_i} x \, d\mu_n(x) - \int_{|x| \leq A_i} x \, d\mu(x) |.
\]

Now we let \( n \to \infty \) to obtain

\[ \limsup_n |EX_n - EX| \leq \left( \frac{1}{i} \cdot \sup_{n \geq 1} E[X_n^2] \right)^{1/2} + E[X; |X| > A_i] \]

And finally we let \( i \to \infty \) to have RHS \( \to 0 \) and get the desired result.

7

First we show that we still need an extra condition so that the weak convergence works.

We compute

\[ E[(n^{-1/2}(X_1 + \cdots + X_n))^4] = \frac{1}{n^2} \left( \sum_{i=1}^{n} E[X_i^4] + 2 \sum_{1 \leq i < j \leq n} E[X_i^2]E[X_j^2] \right) \]
\[ \leq \frac{1}{n^2} (n + n(n-1)) = 1. \]

Use similar techniques as we did in the previous problem using the fact that \( \{X_n\} \) has bounded fourth moments, we claim that \( E[(n^{-1/2}(X_1 + \cdots + X_n))^2] \to E[N(0, \sigma)^2] \), if we assume that \( n^{-1/2}(X_1 + \cdots + X_n) \to N(0, \sigma) \).

As a result, we have to impose an additional condition: \( \lim_{n \to \infty} \frac{1}{3n} \sum_{i=1}^{n} a_i^2 \) exists and is finite. This value is actually \( \sigma^2 \).

Now it’s time for applying Lindeberg-Feller theorem, which states as follows (See Durrett’s book):

\textit{For each } \( n \), let \( X_{n,m}, 1 \leq m \leq n \) \textit{be independent random variables with } \( E[X_{n,m}] = 0 \). \textit{Suppose}
(i) \[
\sum_{m=1}^{n} E[X_{n,m}^2] \to \sigma^2 > 0;
\]
(ii) For all \( \epsilon > 0 \), \( \lim_{n \to \infty} \sum_{m=1}^{n} E[|X_{n,m}|^2] > \epsilon \) = 0.

Then \( S_n = X_{n,1} + \cdots + X_{n,n} \Rightarrow N(0, \sigma^2) \) as \( n \to \infty \).

Let \( X_{n,m} = n^{-1/2}X_m \). The condition (i) is satisfied since we assume \( \lim_{n \to \infty} \frac{1}{3n} \sum_{i=1}^{n} a_i^2 = \sigma^2 \) exists and is finite. For (ii), just note that \( \Pr(|X_{n,m}| > \epsilon) = \Pr(|X_m| > n^{1/2} \epsilon) = 0 \) when \( n > \frac{1}{\epsilon^2} \) for every \( 1 \leq m \leq n \). Thus by Lindeberg-Feller theorem, \( n^{-1/2}(X_1 + \cdots + X_n) \to N(0, \lim_{n \to \infty} \frac{1}{3n} \sum_{i=1}^{n} a_i^2) \).

8

(a) The fact that \( g \geq 0 \) a.e. in \( \mathbb{R}^2 \) is obvious. It is left to show \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \, dx \, dy = 1 \). To see this,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{y} 2f(x)f(y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{x} 2f(y)f(x) \, dy \, dx
\]
\[
= \int_{-\infty}^{\infty} \int_{y}^{\infty} 2f(y)f(x) \, dx \, dy
\]
\[
= \frac{1}{2} (\int_{-\infty}^{\infty} \int_{-\infty}^{y} 2f(x)f(y) \, dx \, dy + \int_{-\infty}^{\infty} \int_{y}^{\infty} 2f(y)f(x) \, dx \, dy)
\]
\[
= \frac{1}{2} \cdot 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)f(y) \, dx \, dy = 1
\]

(b) First we choose \( t \in \mathbb{R} \) so that \( \int_{-\infty}^{t} f(x) \, dx > 0 \) and \( \int_{t}^{\infty} f(x) \, dx > 0 \). We have \( P(X \leq t, Y > t) = 0 \), but both \( P(X \leq t) = \int_{-\infty}^{\infty} \int_{-\infty}^{t} 2f(x)f(y) \, dx \, dy = 2 \int_{-\infty}^{t} f(x) \, dx > 0 \) and \( P(Y > t) = 2 \int_{t}^{\infty} f(y) \, dy > 0 \). This shows \( X \) and \( Y \) are not independent.

9

\[
P(n^{-1/\alpha} \max(X_1, \cdots, X_n) \leq x) = P(\max(X_1, \cdots, X_n) \leq n^{1/\alpha} x)
\]
\[
= P(1 - (n^{1/\alpha} x)^{-\alpha})^n
\]
\[
= p(1 - \frac{x^{-\alpha}}{n})^n \to e^{-x^{-\alpha}}
\]
as \( n \to \infty \). We may define \( F(x) = e^{-x^{-\alpha}} \), and it is continuous and non-decreasing on \( \mathbb{R} \), with \( F(\infty) = 1, F(-\infty) = 0 \). Thus it is the distribution function of some random variable \( X \). So \( P(X > x) = 1 - F(x) = 1 - e^{-x^{-\alpha}} \).
(a) 
\[ f(t) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} e^{itx} \, dx \]
\[ = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \, dx \]
\[ = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(n + \alpha)}{\lambda^{n+\alpha}} \int_0^\infty \frac{\lambda^{n+\alpha}}{\Gamma(n + \alpha)} x^{n+\alpha-1} e^{-\lambda x} \, dx \]
\[ = \sum_{n=0}^{\infty} \frac{(it)^n \lambda^n}{n!} \binom{\alpha}{n} = (1 + \frac{it}{\lambda})^\alpha. \]

(b) \( f_{X_1 + \ldots + X_n}(t) = f_{X_1}(t)^n = (1 + \frac{it}{\lambda})^n. \) That is, it is Gamma\((n, \lambda)\).

(c) We have: \( S_n + X_{n+1} > k, S_n < k. \) Therefore, the probability is
\[ \int_0^k \int_{k-s}^\infty \lambda e^{-\lambda x} \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s} \, dx \, ds \]
\[ = \int_0^k e^{-\lambda(k-s)} \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s} \, ds \]
\[ = \frac{\lambda^n}{(n-1)!} e^{-\lambda k} \int_0^k s^{n-1} \, ds \]
\[ = \frac{(\lambda k)^n}{n!} e^{-\lambda k}. \]