Probability Qualifying Exam Solution, Spring 2009

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December 27, 2012

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(1)

$$\begin{aligned}
E[S_N] &= E[E[S_N|N]] \\
&= E[\sum_{i=0}^{\infty}]1_{\{N=i\}}E[S_i]] \\
&= E[\sum_{i=0}^{\infty}]1_{\{N=i\}}i\mu] \\
&= \mu E[\sum_{i=0}^{\infty}]1_{\{N=i\}}i] = \mu E[N]
\end{aligned}$$

(2)

$$\begin{aligned}
Var[S_N] &= E[E[S_N^2|N]] - (ES_N)^2 \\
&= E[\sum_{i=0}^{\infty} 1_{\{N=i\}}E[S_i^2]] - (\mu E[N])^2 \\
&= E[\sum_{i=0}^{\infty} 1_{\{N=i\}}\big(i\sigma^2 + i(i-1)\mu^2\big)] - \mu^2(E[N])^2 \\
&= \sigma^2 E[N] + \mu^2\big(E[N^2] - E[N]\big) - \mu^2(E[N])^2 \\
&= (\sigma^2 - \mu^2)E[N] + \mu^2 Var[N]
\end{aligned}$$

2

(a)

$$
\begin{aligned}
E[\frac{1}{X+1}] &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{1}{k+1} \\
&= \frac{1}{(n+1)p} \sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} p^{k+1} (1-p)^{n-k} \\
&= \frac{1}{(n+1)p} \Big(\sum_{k=0}^{n+1} \frac{(n+1)!}{(k+1)!(n-k)!} p^{k+1} (1-p)^{n-k} - p^{n+1} \Big) \\
&= \frac{1}{(n+1)p} (1-p^{n+1})
\end{aligned}
$$

(b)

$$
\begin{aligned}
E[\frac{1}{X+1}] &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \cdot \frac{1}{k+1} = \sum_{k=0}^{\infty} \frac{\lambda^{k+1} e^{-\lambda}}{(k+1)!} \\
&= \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1} e^{-\lambda}}{(k+1)!} \\
&= \frac{1}{\lambda} \Big(\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} - e^{-\lambda} \Big) \\
&= \frac{1}{\lambda} (1-e^{-\lambda})
\end{aligned}
$$

3

(a)Given any $A > 0$,

$$
\begin{aligned}
P(|X_n| > A) &= 2 \int_A^{\infty} \frac{n}{\pi(1+n^2x^2)} dx \\
&= \frac{2}{\pi} \tan^{-1}(nx)|_A^{\infty} \\
&= \frac{2}{\pi} (\frac{\pi}{2} - \tan^{-1}(nA)) \to 0 \text{ as } n \to \infty
\end{aligned}
$$

Thus $X_n \to 0$ in pr.

(b)$E[|X_n - 0|] = E[|X_n|] = \frac{2}{\pi}\int_0^\infty \frac{nx}{1+n^2x^2} = \frac{1}{n\pi}\ln(1+n^2x^2)|_0^\infty = \infty$. Thus $X_n \nrightarrow 0$ in L^1.

(c)Given $A > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty} P(|X_n - 0| > A) &= \frac{2}{\pi}\sum_{n=1}^{\infty}(\frac{\pi}{2} - \tan^{-1}(nA)) \\
&\geq \frac{2}{\pi}(\int_0^\infty \frac{\pi}{2} - \tan^{-1}(x)\,dx - \frac{\pi}{2}) \\
&= \frac{2}{\pi}(\int_0^{\frac{\pi}{2}} \tan(x)\,dx - \frac{\pi}{2}) \\
&= \frac{2}{\pi}\big(\ln(\sec\,x)|_0^{\pi/2} - \frac{\pi}{2}\big) = \infty
\end{aligned}$$

By Borel-Cantelli lemma, $P(|X_n - 0| > A\ i.o.) = 1$. This implies the failure of a.s. convergence.

4

Hard.

5

(a)$E[M_n|F_{n-1}] = \lambda_0^{S_{n-1}} E[\lambda_0^{X_n}|F_{n-1}] = \lambda_0^{S_{n-1}} E[\lambda_0^{X_n}] = \lambda_0^{S_{n-1}}$.

(b)Since $E[\lambda_0^{X_1}] = 1$, we let $\lambda_0 = a(\cos\theta + i\sin\theta)$ and we have

$$pa\cos\theta + (1-p)a^{-1}\cos\theta = 1 \tag{1}$$

and

$$pa\sin\theta - (1-p)a^{-1}\sin\theta = 0 \tag{2}$$

by (2), we have either $a = \sqrt{\frac{1-p}{p}}$ or $\sin\theta = 0$. If $a = \sqrt{\frac{1-p}{p}}$, then by (1) we have $\cos\theta = \frac{1}{2\sqrt{p(1-p)}} \geq 1$, and this implies $\cos\theta = 1$ and $p = \frac{1}{2} \Rightarrow a = 1$.

If we start at the assumption that $\sin\theta = 0$, by (1) we have $\cos\theta = 1$ to make $pa\cos\theta + (1-p)a^{-1}\cos\theta$ positive. Hence $pa^2 - a + (1-p) = 0$ and we have $a = 1$ or $a = \frac{1-p}{p}$. Therefore, λ_0 equals 1 or $\frac{1-p}{p}$, and for any n we have

$$E[|M_n|] = E[|X_1|]^n = p|\lambda| + (1-p)|\lambda^{-1}| = p\lambda + (1-p)\lambda^{-1} = 1$$

which shows $\{M_n\}$ is L^1 bounded; martingale convergence theorem tells us the limit exists and is finite a.s.

(c)Let X_1 takes value on 1 and 3 with probability $\frac{1}{2}$ each, and $\lambda_0 = ae^{i\theta}$. We have

$$\frac{1}{2}a\cos\theta + \frac{1}{2}a^3(4\cos^3\theta - 3\cos\theta) = 1 \tag{3}$$

and

$$\frac{1}{2}a\sin\theta + \frac{1}{2}a^3(3\sin\theta - 4\sin^3\theta) = 0 \tag{4}$$

By (4) we have $\sin^2\theta = \frac{3}{4} + \frac{1}{4a^2} = 1 - \cos^\theta$. Substitute $\frac{1}{4} - \frac{1}{4a^2}$ for $\cos^\theta$ in (3) we have $\cos\theta = \frac{-1}{a^3}$. Therefore we have the equation $(\frac{-1}{a^3})^2 = \frac{1}{4} - \frac{1}{4a^2}$, which solves to $a = \sqrt{2}$ since $a > 0$. Therefore, $\cos\theta = \frac{-\sqrt{2}}{4}$, and we choose $\sin\theta = \frac{\sqrt{14}}{4}$.

We find that $E[\lambda_0^{X_1}] = 1$ and $P(|\lambda_0^{X_1}| = \sqrt{2}) = P(|\lambda_0^{X_1}| = 2\sqrt{2}) = \frac{1}{2}$. Thus $|M_n| \geq (\sqrt{2})^n$ a.s., showing that the limit does not exist for a.s. ω.

6

Given $i \in \mathbb{N}$, we may choose A_i s.t. $P(|X| > A_i) < 1/i$ and $F_X(x)$ is continuous at both A_i and $-A_i$, and $A_i \to \infty$ as $i \to \infty$. Therefore,

$$\begin{aligned}
\int_{\{x>0\}} x d\mu &= \lim_{i\to\infty} \int_{\{0<x\leq A_i\}} x d\mu \qquad \text{by monotone convergence theorem} \\
&\leq \limsup_{i\to\infty} |\int_{\{0<x\leq A_i\}} x d\mu - \int_{\{0<x\leq A_i\}} x d\mu_n| + \limsup_{i\to\infty} |\int_{\{0<x\leq A_i\}} x d\mu_n| \\
&\leq \limsup_{i\to\infty} |\int_{\{0<x\leq A_i\}} x d\mu - \int_{\{0<x\leq A_i\}} x d\mu_n| + (\sup_{n\geq 1} E[X_n^2])^{1/2} \\
&\leq 1 + (\sup_{n\geq 1} E[X_n^2])^{1/2},
\end{aligned}$$

if we select $n = n(i)$ above to satisfy $|\int_{\{0<x\leq A_i\}} x d\mu - \int_{\{0<x\leq A_i\}} x d\mu_n| \leq 1$ for each i. This can be done because $X_n \Rightarrow X$ implies $E[f(X_n)] \to E[f(X)]$ for any bounded continuous function. Similarly we can prove the finiteness of $\int_{\{x\leq 0\}} x d\mu$. This proves $X \in L^1$. For the rest part,

$$
\begin{aligned}
\Big|\int x\,d\mu_n(x) - \int x\,d\mu(x)\Big| &\le \Big|\int_{|x|>A_i} x\,d\mu_n(x)\Big| + \Big|\int_{|x|>A_i} x\,d\mu(x)\Big| \\
&+ \Big|\int_{|x|\le A_i} x\,d\mu_n(x) - \int_{|x|\le A_i} x\,d\mu(x)\Big| \\
&\le (\frac{1}{i}\cdot \sup_{n\ge 1} E[X_n^2])^{1/2} + E[X;|X|>A_i] \\
&+ \Big|\int_{|x|\le A_i} x\,d\mu_n(x) - \int_{|x|\le A_i} x\,d\mu(x)\Big|.
\end{aligned}
$$

Now we let $n\to\infty$ to obtain

$$
\limsup_n |EX_n - EX| \le (\frac{1}{i}\cdot \sup_{n\ge 1} E[X_n^2])^{1/2} + E[X;|X|>A_i]
$$

And finally we let $i\to\infty$ to have RHS $\to 0$ and get the desired result.

7

First we show that we still need an extra condition so that the weak convergence works.

We compute

$$
\begin{aligned}
E[\big(n^{-1/2}(X_1+\cdots+X_n)\big)^4] &= \frac{1}{n^2}\big(\sum_{i=1}^n E[X_i^4] + 2\sum_{1\le i<j\le n} E[X_i^2]E[X_j^2]\big) \\
&\le \frac{1}{n^2}(n+n(n-1)) = 1.
\end{aligned}
$$

Use similar techniques as we did in the previous problem using the fact that $\{X_n\}$ has bounded fourth moments, we claim that $E[\big(n^{-1/2}(X_1+\cdots+X_n)\big)^2] \to E[N(0,\sigma)^2]$, if we assume that $n^{-1/2}(X_1+\cdots+X_n)\to N(0,\sigma)$. As a result, we have to impose an additional condition: $\lim_{n\to\infty}\frac{1}{3n}\sum_{i=1}^n a_i^2$ exists and is finite. This value is actually σ^2.

Now it's time for applying Lindeberg-Feller theorem, which states as follows (See Durrett's book):

For each n, let $X_{n,m}, 1\le m\le n$ be independent random variables with $E[X_{n,m}]=0$. Suppose

(i)$\sum_{m=1}^{n} E[X_{n,m}^2] \to \sigma^2 > 0$*;*
(ii)For all $\epsilon > 0, \lim_{n\to\infty} \sum_{m=1}^{n} E[X_{n,m}^2; |X_{n,m}| > \epsilon] = 0$.

Then $S_n = X_{n,1} + \cdots + X_{n,n} \Rightarrow N(0, \sigma^2)$ *as* $n \to \infty$.

Let $X_{n,m} = n^{-1/2} X_m$. The condition (i) is satisfied since we assume $\lim_{n\to\infty} \frac{1}{3n} \sum_{i=1}^{n} a_i^2 = \sigma^2$ exists and is finite. For (ii), just note that $P(|X_{n,m}| > \epsilon) = P(|X_m| > n^{1/2}\epsilon) = 0$ when $n > \frac{1}{\epsilon^2}$ for every $1 \le m \le n$. Thus by Lindeberg-Feller theorem, $n^{-1/2}(X_1 + \cdots + X_n) \to N(0, \lim_{n\to\infty} \frac{1}{3n} \sum_{i=1}^{n} a_i^2)$.

8

(a)The fact that $g \ge 0$ a.e. in $\mathbb{R}^2$ is obvious. It is left to show $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\, dx\, dy = 1$. To see this,

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{y} 2f(x)f(y)\, dx\, dy &= \int_{-\infty}^{\infty} \int_{-\infty}^{x} 2f(y)f(x)\, dy\, dx \\
&= \int_{-\infty}^{\infty} \int_{y}^{\infty} 2f(y)f(x)\, dx\, dy \\
&= \frac{1}{2}\Big(\int_{-\infty}^{\infty} \int_{-\infty}^{y} 2f(x)f(y)\, dx\, dy + \int_{-\infty}^{\infty} \int_{y}^{\infty} 2f(y)f(x)\, dx\, dy\Big) \\
&= \frac{1}{2} \cdot 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)f(y)\, dx\, dy = 1
\end{aligned}$$

(b) First we choose $t \in \mathbb{R}$ so that $\int_{-\infty}^{t} f(x)\, dx > 0$ and $\int_{t}^{\infty} f(x)\, dx > 0$. We have $P(X \le t, Y > t) = 0$, but both $P(X \le t) = \int_{-\infty}^{\infty} \int_{-\infty}^{t} 2f(x)f(y)\, dx\, dy = 2\int_{-\infty}^{t} f(x)\, dx > 0$ and $P(Y > t) = 2\int_{y}^{\infty} f(y)\, dy > 0$. This shows X and Y are not independent.

9

$$\begin{aligned}
P(n^{-1/\alpha} \max(X_1, \cdots, X_n) \le x) &= P(\max(X_1, \cdots, X_n) \le n^{1/\alpha} x) \\
&= P(1 - (n^{1/\alpha} x)^{-\alpha})^n \\
&= P(1 - \frac{x^{-\alpha}}{n})^n \to e^{-x^{-\alpha}}
\end{aligned}$$

as $n \to \infty$. We may define $F(x) = e^{-x^{-\alpha}}$, and it is continuous and non-decreasing on $\mathbb{R}$, with $F(\infty) = 1$, $F(-\infty) = 0$. Thus it is the distribution function of some random variable X. So $P(X > x) = 1 - F(x) = 1 - e^{-x^{-\alpha}}$.

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(a)

$$
\begin{aligned}
f(t) &= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} e^{itx}\, dx \\
&= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \sum_{n=0}^\infty \frac{(itx)^n}{n!}\, dx \\
&= \sum_{n=0}^\infty \frac{(it)^n}{n!} \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(n+\alpha)}{\lambda^{n+\alpha}} \int_0^\infty \frac{\lambda^{n+\alpha}}{\Gamma(n+\alpha)} x^{n+\alpha-1} e^{-\lambda x}\, dx \\
&= \sum_{n=0}^\infty (\frac{it}{\lambda})^n \binom{\alpha}{n} = (1+\frac{it}{\lambda})^\alpha.
\end{aligned}
$$

(b) $f_{X_1+\cdots+X_n}(t) = f_{X_1}(t)^n = (1+\frac{it}{\lambda})^n$. That is, it is $Gamma(n, \lambda)$.

(c) We have: $S_n + X_{n+1} > k$, $S_n < k$. Therefore, the probability is

$$
\begin{aligned}
&\int_0^k \int_{k-s}^\infty \lambda e^{-\lambda x} \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s}\, dx\, ds \\
=&\int_0^k e^{-\lambda(k-s)} \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s}\, ds \\
=&\frac{\lambda^n}{(n-1)!} e^{-\lambda k} \int_0^k s^{n-1}\, ds \\
=&\frac{(\lambda k)^n}{n!} e^{-\lambda k}.
\end{aligned}
$$