Probability Qualifying Exam Solution, Fall 2008

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Let $F_n = \sigma(S_1, \cdots, S_n) = \sigma(X_1, \cdots, X_n)$.

$$\begin{aligned} E[S_n|F_{n-1}] &= E[S_{n-1}|F_{n-1}] + E[X_n|F_{n-1}] \\ &= S_{n-1} + E[X_n|F_{n-1}] \end{aligned}$$

Therefore, $E[X_n|F_{n-1}] = 0$ a.s., and so $E[X_n|F_m] = E[X_n|F_{n-1}]|F_m] = 0$ for all $m < n$. Thus, for $i > j$ we have

$$E[X_iX_j] = E[E[X_iX_j|F_j]] = E[X_jE[X_i|F_j]] = E[X_j \cdot 0] = 0.$$

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(1)To avoid ambiguity, let $X_1 = 0$. To show that $\frac{X_1+\cdots+X_n}{n} \to 0$ in pr, consider

$$\begin{aligned} E[(\frac{X_1 + \cdots + X_n}{n})^2] &= \frac{1}{n^2}\sum_{i=1}^{n} E[X_i^2] = \frac{1}{n^2}\sum_{i=2}^{n}\frac{i}{\log i} \\ &\leq \frac{1}{n^2}(\frac{2}{\log 2} + \sum_{i=3}^{[\sqrt{n}]} i + \sum_{[\sqrt{n}]+1}^{n}\frac{i}{\log i}) \\ &\leq \frac{1}{n^2}\left(\frac{2}{\log 2} + \frac{1}{2}\cdot([\sqrt{n}]^2 + [\sqrt{n}]) + \frac{1}{2}\frac{n^2+n}{\log([\sqrt{n}]+1)}\right) \to 0 \text{ as } n \to \infty \end{aligned}$$

Thus for any $k > 0$, $P(|\frac{X_1+\cdots+X_n}{n}| > k) \leq \frac{1}{k^2}E[(\frac{X_1+\cdots+X_n}{n})^2] \to 0$ as $n \to \infty$.

(2) Note that $\sum_{n=1}^{\infty} P(|X_n| \geq n) = \sum_{n=1}^{\infty}\frac{1}{n\log n} = \infty$, by Borel-Cantelli lemma we have $P(|X_n| \geq n \ i.o.) = 1$. When $|X_n| \geq n$, either $|S_n| > \frac{n}{3}$ or $|S_{n-1}| > \frac{n}{3} > \frac{n-1}{3}$, and thus $P(\frac{|S_n|}{n} > \frac{1}{3} \ i.o.) = 1$. This shows $\frac{S_n}{n} \nrightarrow 0$ a.s.

3

First, we observe that the series

$$\sum_{n=1}^{\infty} \frac{1}{(\sqrt{2}+\epsilon)\log n} \cdot \frac{1}{\sqrt{2\pi}} e^{-(\frac{(\sqrt{2}+\epsilon)^2}{2}\log n)} = \sum_{n=1}^{\infty} \frac{1}{(2+\sqrt{2}\epsilon)\pi} \cdot \frac{1}{\log n} \cdot n^{-(\frac{(\sqrt{2}+\epsilon)^2}{2}}$$

converges when $\epsilon \geq 0$ and diverges when $-\sqrt{2} \leq \epsilon < 0$ by integral test. Since $1 - \Phi(x) \sim \phi(x)/x$ when x is large, by limit comparison test the following series

$$\sum_{n=1}^{\infty} P(|X_n| \geq (\sqrt{2}+\epsilon)\log n)$$

converges when $\epsilon \geq 0$ and diverges when $-\sqrt{2} \leq \epsilon < 0$, which shows $P(|X_n|/\log n \geq (\sqrt{2}+\epsilon)\ i.o.) = 0$ when $\epsilon \geq 0$ and $= 1$ when $-\sqrt{2} \leq \epsilon < 0$. This shows $\limsup_n |X_n|/\log n \geq \sqrt{2}$ a.s.

4

First we claim that $Y_n \in \sigma\{X_1, \cdots, X_n\}$.

$$\begin{aligned} E[Y_n|F_{n-1}] &= E[X_n 1_{\{Y_{n-1}=0\}} + nY_{n-1}|X_n|1_{\{Y_{n-1}\neq 0\}}|F_{n-1}] \\ &= 1_{\{Y_{n-1}=0\}}E[X_n|F_{n-1}] + nY_{n-1}1_{\{Y_{n-1}\neq 0\}E[|X_n|}|F_{n-1}] \\ &= 1_{\{Y_{n-1}=0\}}E[X_n] + nY_{n-1}1_{\{Y_{n-1}\neq 0\}}E[|X_n|] \\ &= 1_{\{Y_{n-1}=0\}} \cdot 0 + Y_{n-1}1_{\{Y_{n-1}\neq 0\}} = Y_{n-1}. \end{aligned}$$

We notice that Y_n is integer valued, and for every nonzero integer k, $P(|Y_n| = k) = 0$ for all $n > |k|$. Besides, $P(Y_n = 0) = P(X_n = 0) = \frac{1}{n} \to 1$ when $n \to \infty$. Thus Y_n converges weakly to 0.

Actually, for any $\epsilon > 0, P(|Y_n| > \epsilon) \leq P(Y_n \neq 0) = P(X_n \neq 0) = \frac{1}{n}$. Let $n \to \infty$ we find that Y_n converges in pr. to 0.

Since $\sum_{n_1}^{\infty} P(|Y_n| > \frac{1}{2}) = \sum_{n=1}^{\infty} P(|Y_n| = 0) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, $P(|Y_n| > \frac{1}{2}\ i.o) = 1$ and hence Y_n does not converge a.s. to 0.

5

(a) It suffices to show that $P(Z > k; X < Y) = P(Z > k)P(X < Y)$ for all $k \in \mathbb{R}$. To see this,

$$
\begin{aligned}
P(Z > k; X < Y) = P(X > k; X < Y) &= \int_k^\infty \int_x^\infty \lambda e^{-\lambda x} \mu e^{-\mu y}\, dy\, dx \\
&= \int_k^\infty \lambda e^{-\lambda x} e^{-\mu x}\, dx \\
&= \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)k} \\
&= e^{-\lambda k} e^{-\mu k} \int_0^\infty \lambda e^{-\lambda x} e^{-\mu x}\, dx \\
&= P(X > k)P(Y > k) \int_0^\infty \int_x^\infty \lambda e^{-\lambda x} \mu e^{-\mu y}\, dy\, dx \\
&= P(Z > k)P(X < Y)
\end{aligned}
$$

(b)

$$
P(X = Z) = P(X \le Y) = \int_0^\infty \int_x^\infty \lambda e^{-\lambda x} \mu e^{-\mu y}\, dy\, dx = \int_0^\infty \lambda e^{-\lambda x} e^{-\mu x} = \frac{\lambda}{\lambda+\mu}
$$

(c) For $k > 0$,

$$
\begin{aligned}
P(U \le k) = P(Y \ge X) &+ \int_0^\infty \int_y^{y+k} \lambda e^{-\lambda x} \mu e^{-\mu y}\, dx\, dy \\
&= \frac{\lambda}{\lambda+\mu} + \int_0^\infty \mu e^{-\mu y}(e^{-\lambda y} - e^{-\lambda(y+k)})\, dy \\
&= \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}(1 - e^{-\lambda k}),
\end{aligned}
$$

and $P(U \le 0) = P(U = 0) = \frac{\lambda}{\lambda+\mu}$; $P(U \le k) = 0$ if $k < 0$.

(d) $V = |X - Y|$; for $k \ge 0$,

$$
\begin{aligned}
P(V \le k) &= \int_0^\infty \int_x^{x+k} \lambda e^{-\lambda x} \mu e^{-\mu y}\, dy\, dx + \int_0^\infty \int_y^{y+k} \lambda e^{-\lambda x} \mu e^{-\mu y}\, dx\, dy \\
&= \int_0^\infty \lambda e^{-\lambda x}(e^{-\mu x} - e^{-\mu(x+k)})\, dx + \int_0^\infty \mu e^{-\mu y}(e^{-\lambda y} - e^{-\lambda(y+k)})\, dy \\
&= \frac{\lambda}{\lambda+\mu}(1 - e^{-\mu k}) + \frac{\mu}{\lambda+\mu}(1 - e^{-\lambda k}),
\end{aligned}
$$

and $P(V \le 0) = 0$.

6

(No brief way to solve it) The MGF of $X_1+\cdots+X_{10}$ is $(\frac{1}{6}(e^t+\cdots+e^{6t}))^{10} = \frac{e^{10t}}{6^{10}}(1+e^t+\cdots+e^{5t})^{10}$. Need detailed calculations.

8

Since $Y_n \Rightarrow Y$, For any $\epsilon > 0$ we may pick $A > 0$ so that $P(|Y| > A) < \frac{\epsilon}{2}$, and hence for all n large enough we have $P(|Y_n| > A) < \epsilon$. Let $B > 0$, we observe that for any bounded continuous function $f(x,y)$, f is uniformly continuous on $\{(x,y) : |x| \leq B, |y| \leq A\}$, and $|f(x_1,y_1) - f(x_2,y_2)| < \epsilon$ whenever $|x_1 - y_1| \leq \delta$. We have

$$
\begin{aligned}
&|\int f(X_n,Y_n)dP - \int f(0,Y)dP| \\
\leq& |\int f(X_n,Y_n)dP - \int f(0,Y_n)dP| + |\int f(0,Y_n)dP - \int f(0,Y)dP| \\
\leq& \int_{\{|X_n|\leq\delta,|Y_n|\leq A\}} |f(X_n,Y_n) - f(0,Y_n)|dP + \int_{\{|X_n|>\delta\}} |f(X_n,Y_n) - f(0,Y_n)|dP \\
+& \int_{\{|Y_n|>A\}} |f(X_n,Y_n) - f(0,Y_n)|dP + |\int f(0,Y_n)dP - \int f(0,Y)dP| \\
\leq& \epsilon + \epsilon K + \epsilon K + |\int f(0,Y_n)dP - \int f(0,Y)dP| \qquad \text{where } K = \sup_{x,y}|f(x,y)| \quad \to \epsilon(1+2k)
\end{aligned}
$$

Since ϵ is arbitrary, we have $\limsup_{n\to\infty} |\int f(X_n,Y_n)dP - \int f(0,Y)dP| = 0$, and this gives us the the desired result.

9

There're several possible constructions. I list two below.

1. Let the joint density of X and Y be $\frac{1}{\pi}e^{-(x^2+y^2)/2}1_{\{x\times y>0\}}$. $P(0 < X+Y < \epsilon) < P(\epsilon < X+Y < 2\epsilon)$ when ϵ is small enough, which impossible for any normal random variable.

2. Let X, Z, W be independent, $X, Z \sim N(0,1)$, W has point mass at $-1, 1$ with probability $1/2$ each. Let $Y = WZ$. It is clear that $X, Y \sim N(0,1)$, while $P(X+Y=0) = 1/2$, so $X+Y$ is not normally distributed and $(X,Y))$ cannot be a Gaussian vector.

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(a) By definition we have $P(X \in B_1)P(Y \in B_2) = P(X \in B_1; Y \in B_2)$ for any $B_1, B_2 \mathscr{B}(\mathbb{R})$.